The key observation is that the given input can be represented as a linear combination of i_0 and multiples of i_0 shifted in time. The process is illustrated in Fig. 5.7; the sum of the three functions shown is i. It is obvious from the graphs of i and i_0 that

$$i = i_0 + 3\mathcal{T}_1(i_0) - 2\mathcal{T}_3(i_0)$$

Now call v the zero-state response due to i; that is,

$$v = \mathcal{Z}_0(i)$$
$$= \mathcal{Z}_0[i_0 + 3\mathcal{T}_1(i_0) - 2\mathcal{T}_3(i_0)]$$

By the linearity of the zero-state response we get

$$v = \mathcal{Z}_0(i_0) + 3\mathcal{Z}_0[\mathcal{T}_1(i_0)] - 2\mathcal{Z}_0[\mathcal{T}_3(i_0)]$$

and by the time-invariance property

$$v = \mathcal{Z}_0(i_0) + 3\mathcal{T}_1[\mathcal{Z}_0(i_0)] - 2\mathcal{T}_3[\mathcal{Z}_0(i_0)]$$

Since

$$v_0 = \mathcal{Z}_0(i_0)$$
$$v = v_0 + 3\mathcal{T}_1(v_0) - 2\mathcal{T}_3(v_0)$$

or

$$v(t) = v_0(t) + 3v_0(t-1) - 2v_0(t-3) \qquad \text{for } t \geq 0$$

Remark The method used to calculate v in terms of v_0 is usually referred to as the *superposition* method. It is fundamental to realize that we have to invoke the time-invariance property and the fact that the zero-state response is a *linear function* of the input.

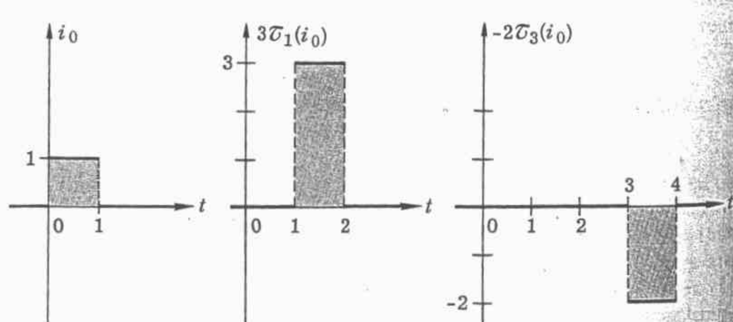


Fig. 5.7 Decomposition of i in terms of shifted pulses.

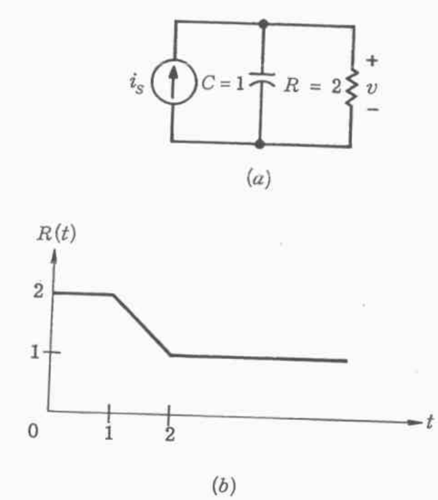


Fig. 5.8 (a) A simple linear RC circuit; (b) time-varying resistor characteristic.

Exercise Consider the familiar linear time-invariant RC circuit shown in Fig. 5.8a; i_s is its input, and v is its response.

a. Calculate and sketch the zero-state response to the following inputs:

$$i_1(t) = \begin{cases} 1 & 0 < t \leq 0.5 \\ 0 & 0.5 < t \end{cases} \qquad i_2(t) = \begin{cases} 3 & 0 < t \leq 0.5 \\ 0 & 0.5 < t \leq 2 \\ -0.5 & 2 < t \leq 2.5 \\ 0 & 2.5 < t \end{cases}$$

b. Suppose now that the resistor is time-varying but still linear. Let its resistance be a function of time as shown in Fig. 5.8b. Suppose we were to calculate the response of this circuit to the input i_1; may we still use the method discussed previously? If not, state briefly why.

6 Impulse Response

The zero-state response of a time-invariant circuit to a *unit* impulse applied at $t = 0$ is called the **impulse response** of a circuit and is denoted by h. More precisely, $h(t)$ is the response at time t of the circuit provided that (1) its input is the unit impulse δ and (2) it is in the *zero state* just prior to the application of the impulse. For convenience in later formulations we shall define h to be zero for $t < 0$. Since the calculation of impulse response is of great importance to electrical engineers, we shall present three methods.

First method We approximate the impulse by the pulse function p_Δ. In order to obtain a first acquaintance with the impulse response, let us calculate the impulse response of the parallel RC circuit shown in Fig. 6.1. The input to the circuit is the current source i_s, and the response is the output voltage v. Since the impulse response is defined to be the zero-state response to δ, the impulse response is the solution of the differential equation

$$C\frac{dv}{dt} + Gv = \delta(t) \tag{6.1}$$

with

$$v(0-) = 0 \tag{6.2}$$

where the symbol $0-$ designates the time immediately before $t = 0$.

We have to distinguish between $0-$ and $0+$ because of the presence of the impulse on the right-hand side of (6.1). At time $t = 0$ an infinitely large current goes through the circuit for an infinitesimal interval of time. The situation is analogous to the golf ball sitting on the tee and being hit by the club at $t = 0$; it is obviously of great importance to distinguish between the velocity of the ball at $0-$ just prior to being hit, and its velocity at $0+$ just after being hit.

Equation (6.2) states that the circuit is in the zero state just prior to the application of the input. In order to solve (6.1) we run into some difficulties since, strictly speaking, δ is *not* a function. Therefore, the solution will be obtained by approximating unit impulse δ by the pulse function p_Δ, computing the resulting solution, and then letting $\Delta \to 0$. Recall that p_Δ is defined by

$$p_\Delta(t) = \begin{cases} 0 & \text{for } t < 0 \\ \dfrac{1}{\Delta} & \text{for } 0 < t < \Delta \\ 0 & \text{for } \Delta < t \end{cases}$$

and it is plotted in Fig. 6.2. The first step is to solve for h_Δ, the zero-state

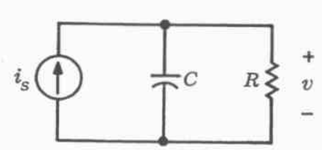


Fig. 6.1 Linear time-invariant RC circuit.

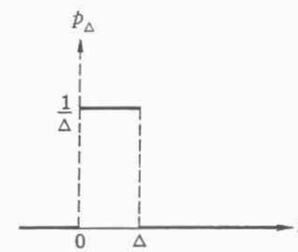


Fig. 6.2 Pulse function $p_\Delta(\cdot)$.

response of the RC circuit to p_Δ, where Δ is chosen to be much smaller than the time constant RC. The waveform h_Δ is the solution of

$$C\frac{dh_\Delta}{dt} + \frac{1}{R}h_\Delta = \frac{1}{\Delta} \qquad 0 < t < \Delta \tag{6.3a}$$

$$C\frac{dh_\Delta}{dt} + \frac{1}{R}h_\Delta = 0 \qquad t > \Delta \tag{6.3b}$$

with $h_\Delta(0) = 0$. Clearly, $1/\Delta$ is a constant; hence from (6.3a)

$$h_\Delta(t) = \frac{R}{\Delta}(1 - \epsilon^{-t/RC}) \qquad 0 < t < \Delta \tag{6.4a}$$

and it is the zero-state response due to a step $(1/\Delta)u(t)$. From (6.3b), h_Δ for $t > \Delta$ is the zero-input response that starts from $h_\Delta(\Delta)$ at $t = \Delta$; thus

$$h_\Delta(t) = h_\Delta(\Delta)\epsilon^{-(t-\Delta)/RC} \qquad t > \Delta \tag{6.4b}$$

The total response h_Δ from (6.4a) and (6.4b) is shown on Fig. 6.3a. From (6.4a)

$$h_\Delta(\Delta) = \frac{R}{\Delta}(1 - \epsilon^{-\Delta/RC})$$

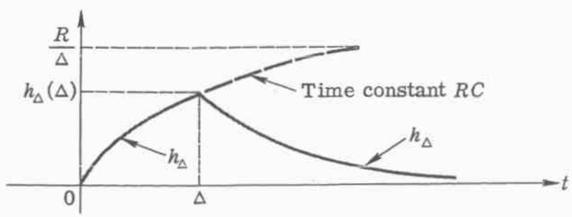
Since Δ is much smaller than RC, using

$$\epsilon^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

we obtain

$$h_\Delta(\Delta) = \frac{R}{\Delta}\left[\frac{\Delta}{RC} - \frac{1}{2!}\left(\frac{\Delta}{RC}\right)^2 + \cdots\right]$$
$$= \frac{1}{C}\left[1 - \frac{1}{2!}\left(\frac{\Delta}{RC}\right) + \cdots\right]$$

Similarly, from (6.4a) for Δ very small and $0 < t < \Delta$, expanding the exponential function, we obtain



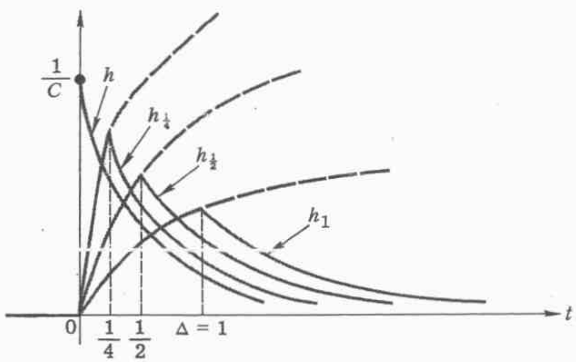


Fig. 6.3 (a) Zero-state response of p_Δ; (b) the responses as $\Delta \to 0$.

$$h_\Delta(t) = \frac{1}{C}\frac{t}{\Delta} + \cdots \qquad 0 < t < \Delta$$

Note that the slope of the curve h_Δ over $(0,\Delta)$ is $1/C\Delta$. This slope is very large since Δ is small. As $\Delta \to 0$, the curve h_Δ over $(0,\Delta)$ becomes steeper and steeper, and $h_\Delta(\Delta) \to 1/C$. In the limit, h_Δ jumps from 0 to $1/C$ at the instant $t = 0$. For $t > 0$, we obtain, from (6.4b),

$$h_\Delta(t) \to \frac{1}{C}\epsilon^{-t/RC}$$

As Δ approaches zero, h_Δ approaches the impulse response h as shown in Fig. 6.3b. Recalling that by convention we set $h(t) = 0$ for $t < 0$, we can therefore write

$$h(t) = u(t)\frac{1}{C}\epsilon^{-t/RC} \qquad \text{for all } t \tag{6.5}$$

The impulse response h is shown in Fig. 6.4.

The above calculation of h calls for two remarks.

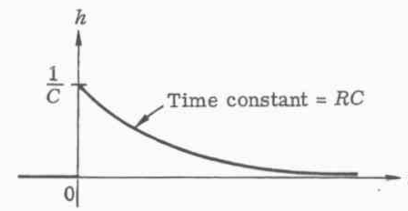


Fig. 6.4 Impulse response of the RC circuit of Fig. 6.1.

Remarks

1. Our purpose in calculating the impulse response in this manner has been to exhibit the fact that it is a straightforward procedure; it requires only the approximation of δ by a suitable pulse, here p_Δ. The only requirements that p_Δ must satisfy are that it be zero outside the interval $(0,\Delta)$ and the area under p_Δ be equal to 1; that is,

$$\int_0^\Delta p_\Delta(t)\,dt = 1$$

It is a fact that the shape of p_Δ is irrelevant; therefore, we choose a shape that requires the least amount of work. We might very well have chosen a triangular pulse as shown in Fig. 6.5. Observe that the maximum amplitude of the triangular pulse is now $2/\Delta$; this is required in order that the area under the pulse be unity for all $\Delta > 0$.

2. Since $\delta(t) = 0$ for $t > 0$ (that is, the input is identically zero for $t > 0$), it follows that the impulse response $h(t)$ is, for $t > 0$, identical to a particular zero-input response. We shall use this fact later.

Relation between impulse response and step response

We wish now to establish a very important relation between the step response and the impulse response of a linear time-invariant circuit. More precisely we wish to show that the following is true:

The impulse response of a linear time-invariant circuit is the time derivative of its step response.

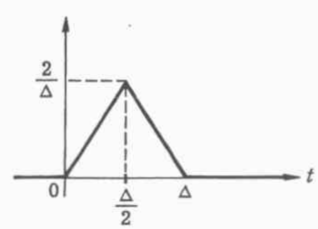


Fig. 6.5 A triangular pulse can also be used to approximate the impulse.

Symbolically,

(6.6) $$h = \frac{ds}{dt} \qquad \text{or equivalently} \qquad s(t) = \int_{-\infty}^{t} h(t')\,dt'$$

We prove this important statement by approximating the impulse by the pulse function p_Δ. Let h_Δ be the zero-state response to the input p_Δ; that is,

$$h_\Delta \triangleq \mathscr{Z}_0(p_\Delta)$$

As $\Delta \to 0$, the pulse function p_Δ approaches δ, the unit impulse, and h_Δ, the zero-state response to the pulse input, approaches the impulse response h. Now consider p_Δ as a superposition of a step and a delayed step as shown in Fig. 6.6. Thus,

$$p_\Delta = \frac{1}{\Delta}[u(t) - u(t-\Delta)] = \frac{1}{\Delta}u + \frac{-1}{\Delta}\mathscr{T}_\Delta u$$

By the linearity of the zero-state response, we have

$$\mathscr{Z}_0(p_\Delta) = \mathscr{Z}_0\left(\frac{1}{\Delta}u + \frac{-1}{\Delta}\mathscr{T}_\Delta u\right)$$

(6.7) $$= \frac{1}{\Delta}\mathscr{Z}_0(u) + \frac{-1}{\Delta}\mathscr{Z}_0(\mathscr{T}_\Delta u)$$

Since the circuit is linear and time-invariant, the $\mathscr{Z}_0$ operator and the shift operator commute; thus,

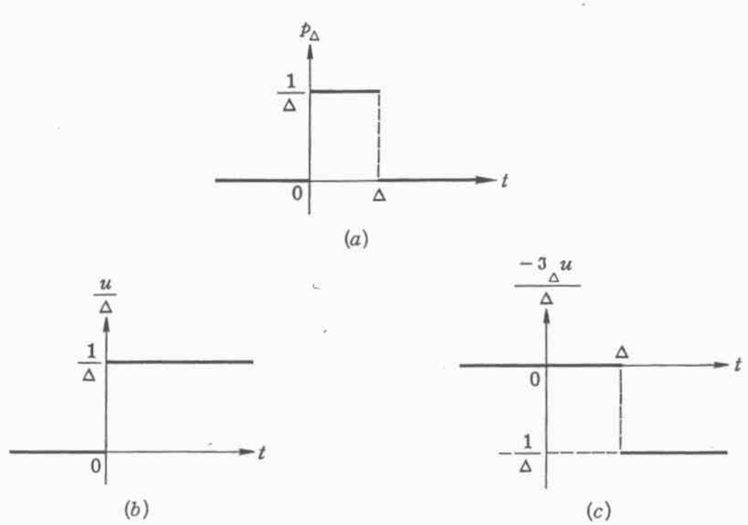


Fig. 6.6 The pulse function p_Δ in (a) can be considered as the sum of a step function in (b) and a delayed step function in (c).

(6.8) $$\mathscr{Z}_0(\mathscr{T}_\Delta u) = \mathscr{T}_\Delta \mathscr{Z}_0(u)$$

Let us denote the step response by

$$s \triangleq \mathscr{Z}_0(u)$$

Equations (6.7) and (6.8) can be combined to yield successively

$$h_\Delta \triangleq \mathscr{Z}_0(p_\Delta) = \frac{1}{\Delta}s - \frac{1}{\Delta}\mathscr{T}_\Delta s$$

$$h_\Delta(t) = \frac{1}{\Delta}s(t) - \frac{1}{\Delta}s(t-\Delta)$$

$$= \frac{s(t) - s(t-\Delta)}{\Delta} \qquad \text{for all } t$$

Now with $\Delta \to 0$, the right-hand side becomes the derivative; hence

$$\lim_{\Delta\to 0} h_\Delta(t) = h(t) = \frac{ds}{dt}$$

Remark The two equations in (6.6) do not hold for linear *time-varying* circuits; this should be expected since time invariance is used in a key step of the derivation. Thus, for linear *time-varying* circuits the time derivative of the step response is *not* the impulse response.

Second method We use $h = ds/dt$. Again considering the parallel RC circuit of Fig. 6.1, we recall that its step response s is given by

$$s(t) = u(t)R(1 - \epsilon^{-(1/RC)t})$$

If we consider the right-hand side as a product of two functions and use the rule of differentiation $(uv)' = u'v + uv'$, we obtain the impulse response

$$h(t) = \delta(t)R(1 - \epsilon^{-(1/RC)t}) + \frac{1}{C}u(t)\epsilon^{-(1/RC)t}$$

The first term is identically zero because for $t \neq 0$, $\delta(t) = 0$, and for $t = 0$, $1 - \epsilon^{-(1/RC)t} = 0$.† Therefore,

$$h(t) = \frac{1}{C}u(t)\epsilon^{-(1/RC)t}$$

This result, of course, checks with the previously obtained result in (6.5).

Third method We use the differential equation directly. We propose to show that h defined by

$$h(t) = \frac{1}{C}u(t)\epsilon^{-t/RC} \qquad \text{for all } t$$

† As a rule replace right away expressions like $f(t)\delta(t)$ by $f(0)\delta(t)$, and expressions like $f(t)\delta(t-\tau)$ by $f(\tau)\delta(t-\tau)$.

is the solution to the differential equation

(6.9) $$C\frac{d}{dt}(v) + Gv = \delta \qquad \text{with } v(0-) = 0$$

In order not to prejudice the case, let us call y the solution to (6.9). Thus, we propose to show that $y = h$. Since $\delta(t) = 0$ for $t > 0$ and y is the solution of (6.9), we must have

(6.10) $$y(t) = y(0+)\epsilon^{-t/RC} \qquad \text{for } t > 0$$

This is shown in Fig. 6.7*a*. Since $\delta(t) = 0$ for $t < 0$ and the circuit is in the zero state at time $0-$, we must also have

(6.11) $$y(t) = 0 \qquad \text{for } t < 0$$

This is shown in Fig. 6.7*b*. Combining (6.10) and (6.11), we conclude that

(6.12) $$y(t) = u(t)y(0+)\epsilon^{-t/RC} \qquad \text{for all } t$$

It remains to calculate $y(0+)$, that is, the magnitude of the jump in the curve y at $t = 0$. In order to do this we use the known fact that

$$\delta(t) = \frac{du(t)}{dt}$$

From (6.12) and by considering the right-hand side as a product of functions, we obtain

$$\frac{dy}{dt}(t) = \delta(t)y(0+)\epsilon^{-t/RC} + u(t)y(0+)\frac{-1}{RC}\epsilon^{-t/RC}$$

In the first term, since $\delta(t)$ is zero everywhere except at $t = 0$, we may set t to zero in the factor of $\delta(t)$; thus

$$\frac{dy}{dt}(t) = \delta(t)y(0+) + u(t)y(0+)\frac{-1}{RC}\epsilon^{-t/RC}$$

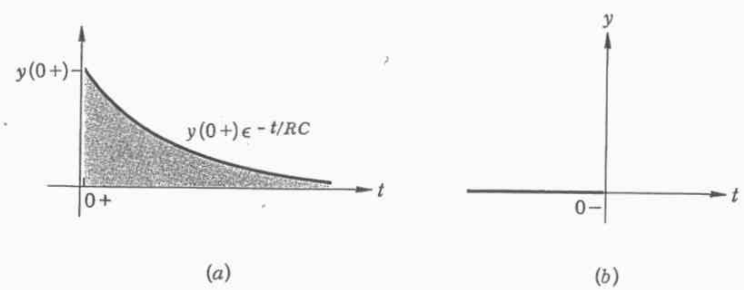


Fig. 6.7 Impulse response for the parallel RC circuit. (*a*) $y(t)$ for $t > 0$; (*b*) $y(t)$ for $t < 0$.

Substituting in (6.9), we obtain

$$\delta(t)Cy(0+) - u(t)y(0+)G\epsilon^{-t/RC} + Gu(t)y(0+)\epsilon^{-t/RC} = \delta(t)$$

After cancellation the only term that remains on the left-hand side is $Cy(0+)\delta(t)$; since it must balance the term $\delta(t)$ in the right-hand side, we obtain $y(0+)C = 1$; equivalently,

$$y(0+) = \frac{1}{C}$$

Inserting this value of $y(0+)$ into (6.12), we conclude that the solution of (6.9) is actually h, the impulse response calculated previously.

Remark We have just shown that the solution of the differential equation

$$C\frac{d}{dt}(v) + Gv = \delta \qquad \text{with } v(0-) = 0$$

for $t > 0$ is identical with the solution of

(6.13) $$C\frac{d}{dt}(v) + Gv = 0 \qquad \text{with } v(0+) = \frac{1}{C}$$

for $t > 0$. This can be seen by integrating both sides of (6.9) from $t = 0-$ to $t = 0+$ to obtain

$$Cv(0+) - Cv(0-) + G\int_{0-}^{0+} v(t')\,dt' = 1$$

Since v is finite, $G\int_{0-}^{0+} v(t')\,dt' = 0$, and since $v(0-) = 0$, we obtain

$$v(0+) = \frac{1}{C}$$

In Eq. (6.13) the effect of the impulse at $t = 0$ has been taken care of by the initial condition at $t = 0+$.

7 Step and Impulse Responses for Simple Circuits

Example 1 Let us calculate the impulse response and the step response of the RL circuit shown in Fig. 7.1. The series connection of the linear time-invariant resistor and inductor is driven by a voltage source. As far as the impulse response is concerned, the differential equation for the current i is

(7.1) $$L\frac{di}{dt} + Ri = \delta \qquad i(0-) = 0$$

If we confine our attention to the values of $t > 0$, this problem is equiva-

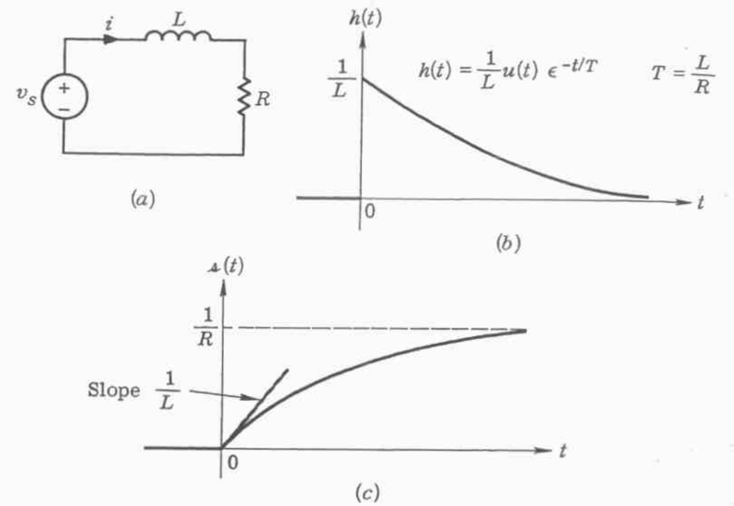


Fig. 7.1 (a) Linear time-invariant RL circuit, v_s is the input and i is the response; (b) impulse response; (c) step response.

lent to that of the same circuit with no voltage source but with the initial condition $i(0+) = 1/L$; that is, for $t > 0$,

$$L\frac{di}{dt} + Ri = 0 \qquad i(0+) = \frac{1}{L} \tag{7.2}$$

The solution is

$$i(t) = h(t) = \frac{1}{L}u(t)\epsilon^{-(R/L)t} \tag{7.3}$$

The step response can be obtained either from integration of (7.3) or directly from the differential equation

$$\mathfrak{s}(t) = \frac{1}{R}u(t)(1 - \epsilon^{-(R/L)t}) \tag{7.4}$$

The physical explanation of the step response of the series RL circuit is now given. As the step of voltage is applied to the circuit, that is, at $0+$, the current in the circuit remains zero because, as we noted earlier, the current through an inductor cannot change instantaneously unless there is an infinitely large voltage across it. Since the current is zero, the voltage across the resistor must be zero. Therefore, at $0+$ all the voltage of the voltage source appears across the inductor; in fact $\left.\frac{di}{dt}\right|_{0+} = 1/L$. As time increases, the current increases monotonically, and after a very long time, the current becomes practically constant. Thus, for large t, $di/dt \approx 0$; that is, the voltage across the inductor is zero, and all the voltage of the source is across the resistor. Therefore, the current is approximately $1/R$. In the limit we reach what is called the *steady state* and $i = 1/R$. We conclude that the inductor behaves as a short circuit in the steady state for a step-voltage input.

Example 2 Consider the circuit in Fig. 7.2, where the series connection of a linear time-invariant resistor R and a capacitor C is driven by a voltage source. The current through the resistor is the response of interest, and the problem is to find the impulse and step responses. The equation for the current i is given by writing KVL for the loop; thus

$$\frac{1}{C}\int_0^t i(t')\,dt' + Ri(t) = v_s(t) \tag{7.5}$$

Let us use the charge on the capacitor as the variable; then (7.5) becomes

$$\frac{q}{C} + R\frac{dq}{dt} = v_s(t) \tag{7.6}$$

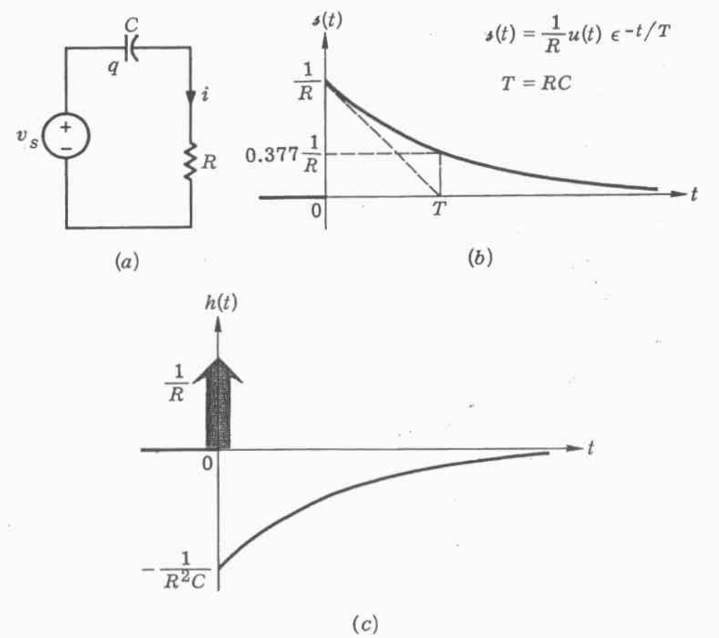


Fig. 7.2 (a) Linear time-invariant RC circuit; v_s is the input and i is the response; (b) step response; (c) impulse response.

Table 4.1 Step and Impulse Responses for Simple Linear Time-invariant Circuits

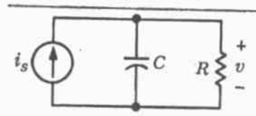
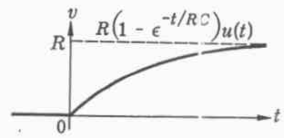
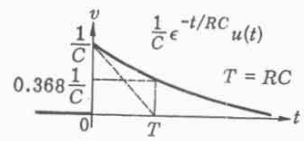
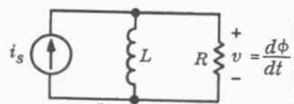
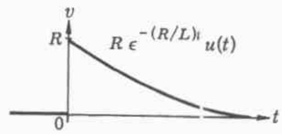
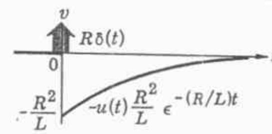
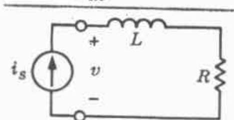
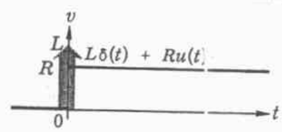
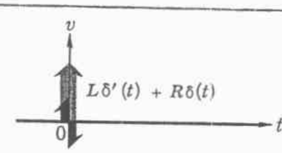
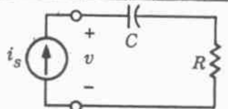
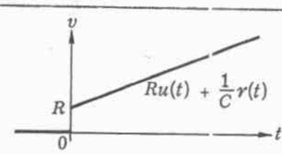
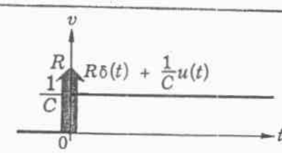
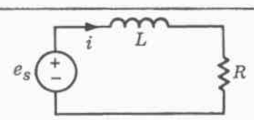
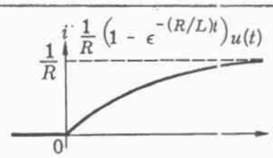
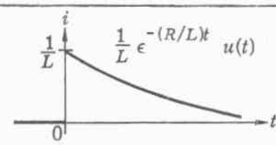
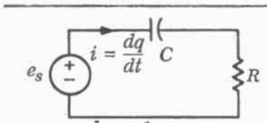
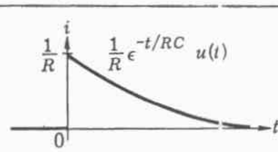
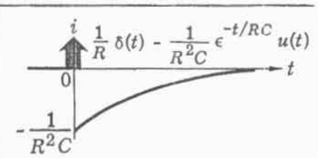
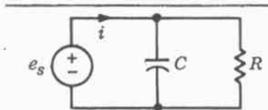
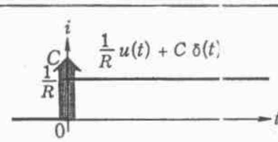
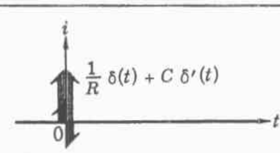
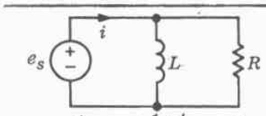
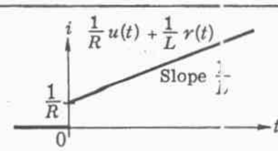
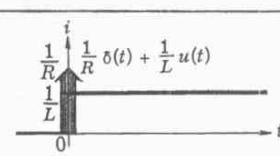
i_s(input) v(response)	$\mathscr{A}(t)$	$h(t)$
 $C\frac{d}{dt}v + \frac{1}{R}v = i_s$		
 $\frac{1}{R}\frac{d}{dt}\phi + \frac{1}{L}\phi = i_s$		
 $v = Ri_s + L\frac{di_s}{dt}$		
 $v = Ri_s + \frac{1}{C}\int_0^t i_s(t')\,dt'$		

Table 4.1 Step and Impulse Responses for Simple Linear Time-invariant Circuits *(Continued)*

e_s(input) i(response)	$\mathscr{A}(t)$	$h(t)$
 $L\frac{di}{dt} + Ri = e_s$		
 $R\frac{d}{dt}q + \frac{1}{C}q = e_s$		
 $i = C\frac{de_s}{dt} + \frac{1}{R}e_s$		
 $i = \frac{1}{R}e_s + \frac{1}{L}\int_0^t e_s(t')\,dt'$		

Since we have to find the step and impulse responses, the initial condition is $q(0-) = 0$. If v_s is a unit step, (7.6) gives

$$q_s(t) = u(t)C(1 - \epsilon^{-t/RC})$$

and by differentiation, the step response for the current is

$$i_s(t) = s(t) = \frac{1}{R}u(t)\epsilon^{-t/RC}$$

If v_s is a unit impulse, (7.6) gives

$$q_\delta(t) = \frac{1}{R}\,u(t)\epsilon^{-t/RC}$$

and, by differentiation, the impulse response for the current is

$$i_\delta(t) = h(t) = \frac{1}{R}\,\delta(t) - \frac{1}{R^2C}\,u(t)\epsilon^{-t/RC}$$

We observe that in response to a step, the current is discontinuous at $t = 0$; $i_s(0+) = 1/R$ as we expect, since at $t = 0$ there is no charge (hence no voltage) on the capacitor. In response to an impulse, the current includes an impulse of value $1/R$, and, for $t > 0$, the capacitor discharges through the resistor.

The step and impulse responses for simple first-order linear time-invariant circuits are tabulated in Table 4.1.

8 Time-varying Circuits and Nonlinear Circuits

Up to this point we have analyzed almost exclusively linear time-invariant circuits. We have studied the implications of the linearity and of the time invariance of element characteristics as far as the relation between input and output is concerned. In this section we shall first summarize the main implications of linearity and of time invariance of element characteristics. Next we shall consider examples of circuits with nonlinear and of time-varying elements to demonstrate that without linearity and time invariance these main implications are no longer true.

In our study of first-order circuits we have seen that if the circuits are *linear* (time-invariant or time-varying), then

1. The zero-input response is a linear function of the initial state.
2. The zero-state response is a linear function of the input.
3. The complete response is the sum of the zero-input response and of the zero-state response.

We have also seen that if the circuit is *linear* and *time-invariant,* then

1. $\mathfrak{Z}_0[\mathfrak{T}_\tau(i)] = \mathfrak{T}_\tau[\mathfrak{Z}_0(i)] \qquad \tau \geq 0$

which means that the zero-state response (starting in the zero state at time zero) to the shifted input is equal to the shift of the zero-state response (starting also in the zero state at time zero) to the original input.

2. The impulse response is the derivative of the step response.

For *time-varying* circuits and *nonlinear* circuits the analysis problem is in general difficult. Furthermore there exists no general method of analysis except numerical integration of the differential equations. Consequently, we shall give only simple examples to point out techniques that may be useful in simple cases. Our main emphasis is, however, to demonstrate certain properties of the solutions.

Example 1 Consider the parallel RC circuit of Fig. 8.1, where the capacitor is linear and time-invariant with $C = 1$ farad and the initial voltage at $t = 0$ is 1 volt. The zero-input responses are to be determined for the following types of resistor:

a. A linear time-invariant resistor with $R = 1$ ohm

b. A linear time-varying resistor with $R(t) = 1/(1 + 0.5 \cos t)$ ohm

c. A nonlinear time-invariant resistor having a characteristic $i_R = v_R^2$

Solution *a.* The solution has been discussed before and is of the form

$$v(t) = \epsilon^{-t} \qquad t \geq 0$$

b. The differential equation is given by

$$\frac{dv}{dt} + (1 + 0.5 \cos t)v = 0 \qquad t \geq 0$$

and

$$v(0) = 1$$

The equation can be put in the following form

$$\frac{dv}{v} = -(1 + 0.5 \cos t)\, dt$$

Integrating the right-hand side from zero to t and the left-hand side from $v(0) = 1$ to $v(t)$, we obtain

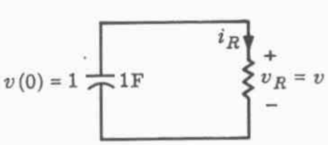


Fig. 8.1 Illustration of the zero-input responses of a simple RC circuit.

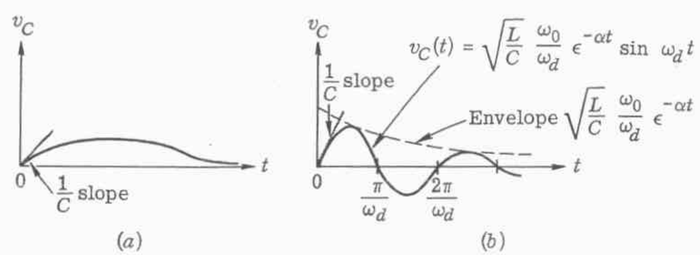


Fig. 2.4 Step responses for the capacitor voltage of the parallel RLC circuit.

increases, and the current will flow in both the resistor and inductor. After a long time the circuit reaches a steady state; that is,

$$\frac{di_L}{dt} = 0 \qquad \frac{d^2 i_L}{dt^2} = 0$$

Hence, according to Eq. (2.2), all current from the source goes through the inductor. Therefore, the voltage across the parallel circuit is zero because the current in the resistor is zero. At $t = \infty$ *the inductor acts as a short circuit to a constant current source.* The currents through the capacitor, resistor, and inductor are plotted in Fig. 2.5 for the overdamped case ($Q < ½$).

Exercise For the parallel RLC circuit with $R = 1$ ohm, $C = 1$ farad, and $L = 1$ henry, determine the currents in the inductor, the capacitor, and the resistor as a result of an input step of current of 1 amp. The circuit is in the zero state at $t = 0-$. Plot the waveforms.

2.2 Impulse Response

We now calculate the impulse response for the parallel RLC circuit. By definition, the input is a unit impulse, and the circuit is in the zero state at $0-$; hence, the impulse response i_L is the solution of

$$LC\frac{d^2 i_L}{dt^2} + LG\frac{di_L}{dt} + i_L = \delta(t) \tag{2.24}$$

$$i_L(0-) = 0 \tag{2.25}$$

$$\frac{di_L}{dt}(0-) = 0 \tag{2.26}$$

Since the computation and physical understanding of the impulse response are of great importance in circuit theory, we shall again present several methods and interpretations, treating only the underdamped case, that is, the circuit with complex natural frequencies.

First method We use the differential equation directly. Since the impulse function $\delta(t)$ is identically zero for $t > 0$, we can consider the impulse response as a zero-input response starting at $t = 0+$. The impulse at $t = 0$ creates an initial condition at $t = 0+$, and the impulse response for $t > 0$ is essentially the zero-input response due to that initial condition. The problem then is to determine this initial condition. Let us integrate both sides of Eq. (2.24) from $t = 0-$ to $t = 0+$. We obtain

$$LC\frac{di_L}{dt}(0+) - LC\frac{di_L}{dt}(0-) + LGi_L(0+) - LGi_L(0-) + \int_{0-}^{0+} i_L(t')\,dt' = 1 \tag{2.27}$$

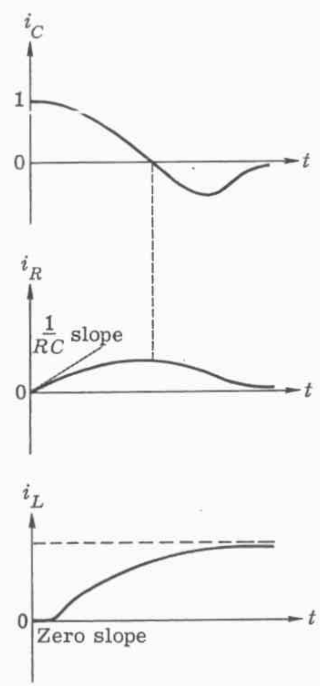


Fig. 2.5 Plots of i_C, i_R, and i_L due to a step-current input for the parallel RLC circuit (overdamped case, $Q < ½$).

where the right-hand side is obtained by using the fact that

$$\int_{0-}^{0+} \delta(t')\,dt' = 1$$

We know that i_L cannot jump at $t = 0$, or equivalently, that i_L is a continuous function; that is,

$$\int_{0-}^{0+} i_L(t')\,dt' = 0 \text{ and } i_L(0+) = i_L(0-)$$

If it were not continuous, di_L/dt would contain an impulse, d^2i_L/dt^2 would contain a doublet, and (2.24) could never be satisfied since there is no doublet on the right-hand side. From (2.27) we obtain

(2.28) $$\frac{di_L}{dt}(0+) = \frac{di_L}{dt}(0-) + \frac{1}{LC} = \frac{1}{LC}$$

As far as $t > 0$ is concerned, the nonhomogeneous differential equation (2.24), with the initial condition given in (2.25) and (2.26), is equivalent to

(2.29) $$LC\frac{d^2i_L}{dt^2} + LG\frac{di_L}{dt} + i_L = 0$$

with

(2.30) $$i_L(0+) = 0$$

and

(2.31) $$\frac{di_L}{dt}(0+) = \frac{1}{LC}$$

For $t \le 0$, clearly, $i_L(t)$ is zero. The solution of the above is therefore

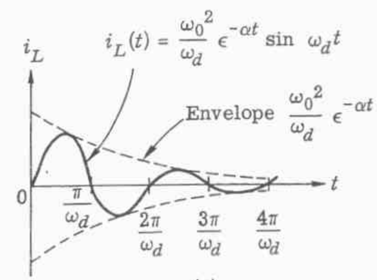
(2.32) $$i_L(t) = u(t)\frac{\omega_0^2}{\omega_d}\epsilon^{-\alpha t}\sin\omega_d t$$

The waveform is shown in Fig. 2.6*a*. Note that (2.32) can also be obtained from the zero-input response of (1.24) for a given initial state $I_0 = 0$ and $V_0 = 1/C$.

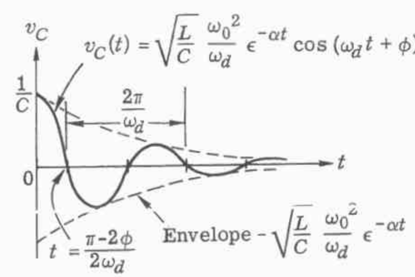
Remark Consider the parallel connection of the capacitor and the current source i_s. In Chap. 2 we showed that the parallel connection is equivalent to the series connection of the same capacitor and a voltage source v_s, where

$$v_s(t) = \frac{1}{C}\int_{0-}^{t} i_s(t')\,dt' \qquad t \ge 0$$

Thus, for an impulse current source, the equivalent voltage source is $(1/C)u(t)$. For $t < 0$, the voltage source is identically zero, and for $t > 0$, the voltage source is a constant $1/C$. The series connection of an uncharged capacitor and a constant voltage source is equivalent to a charged capacitor with initial voltage $1/C$. Therefore, the impulse response of a



(*a*)



(*b*)

Fig. 2.6 Impulse response of the parallel *RLC* circuit for the underdamped case ($Q > ½$).

parallel *RLC* circuit due to a current impulse in parallel is the same as a zero-input response with $v_C(0+) = 1/C$. These equivalences are illustrated in Fig. 2.7.

Direct substitution Let us verify by direct substitution into Eqs. (2.24) to (2.26) that (2.32) is the solution. This is a worthwhile exercise for getting familiar with manipulations involving impulses. First, i_L as given by (2.32) clearly satisfies the initial conditions of (2.25) and (2.26); that is, $i_L(0-) = 0$ and $(di_L/dt)(0-) = 0$. It remains for us to show that (2.32) satisfies the differential equation (2.24). Differentiating (2.32), we obtain

(2.33) $$\frac{di_L}{dt} = \delta(t)\left(\frac{\omega_0^2}{\omega_d}\epsilon^{-\alpha t}\sin\omega_d t\right) + \frac{u(t)\omega_0^3}{\omega_d}\epsilon^{-\alpha t}\cos(\omega_d t + \phi)$$

Now the first term is of the form $\delta(t)f(t)$. Since $\delta(t)$ is zero whenever $t \neq 0$, we may set $t = 0$ in the factor and obtain $\delta(t)f(0)$; however $f(0) = 0$. Hence the first term of (2.33) disappears, and

(2.34) $$\frac{di_L}{dt} = \frac{u(t)\omega_0^3}{\omega_d}\epsilon^{-\alpha t}\cos(\omega_d t + \phi)$$

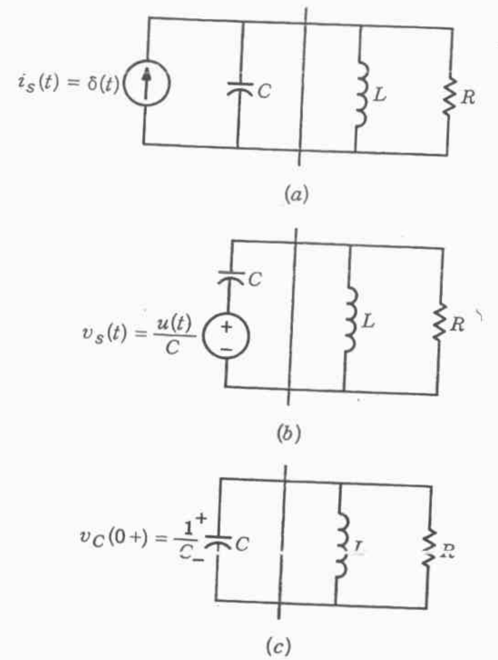


Fig. 2.7 The problem of determining the impulse response of a parallel RLC circuit is reduced to that of determining the zero-input response of an RLC circuit. Note that the parallel connection of the capacitor and the impulse current source in (a) is reduced to the series connection of the capacitor and a step voltage source in (b) and is further reduced to a charged capacitor in (c).

Differentiating again, we obtain

$$\frac{d^2 i_L}{dt^2} = \delta(t)\frac{\omega_0^3}{\omega_d}\cos\phi - u(t)\frac{\omega_0^4}{\omega_d}\epsilon^{-\alpha t}\sin(\omega_d t + 2\phi)$$

(2.35) $$= \omega_0^2\delta(t) - u(t)\frac{\omega_0^4}{\omega_d}\epsilon^{-\alpha t}[\sin(\omega_d t + \phi)\cos\phi + \cos(\omega_d t + \phi)\sin\phi]$$

Substituting Eqs. (2.32), (2.34), and (2.35) in (2.24), which is rewritten below in terms of ω_0 and α,

$$\frac{1}{\omega_0^2}\frac{di_L^2}{dt^2} + \frac{2\alpha}{\omega_0^2}\frac{di_L}{dt} + i_L = \delta(t)$$

we shall see that the left-hand side is equal to $\delta(t)$ as it should be. Thus,

we have verified by direct substitution that (2.32) is the impulse response of the parallel RLC circuit.

Exercise Show that the impulse response for the capacitor voltage of the parallel RLC circuit is

(2.36) $$v_C(t) = u(t)\sqrt{\frac{L}{C}}\frac{\omega_0^2}{\omega_d}\epsilon^{-\alpha t}\cos(\omega_d t + \phi)$$

The waveform is shown in Fig. 2.6*b*.

Second method We use the relation between the impulse response and the step response. This method is applicable only to circuits with linear time-invariant elements for it is only for such circuits that the impulse response is the derivative of the step response.

Exercise Show that the impulse responses for i_L in Eq. (2.32) and v_C in Eq. (2.36) are obtainable by differentiating the step response for i_L in Eq. (2.21) and v_C in Eq. (2.23).

Physical interpretation Let us use the pulse input $i_s(t) = p_\Delta(t)$ as shown in Fig. 2.8*a* to explain the behavior of all the branch currents and voltages in the parallel RLC circuit. Remember that as $\Delta \to 0$, pulse p_Δ approaches an impulse, and the response approaches the impulse response. To start with, we assume Δ is finite and positive, but very small. From the discussion of the step

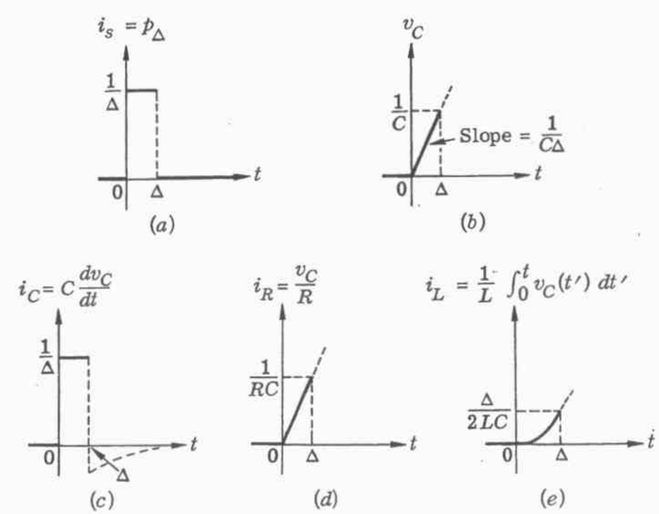


Fig. 2.8 Physical explanation of impulse response of a parallel RLC circuit; p_Δ is the input pulse; the resulting v_C, i_C, i_R, and i_L are shown.

response, we learned that at $t = 0+$ all current from the source goes into the capacitor; that is, $i_C(0+) = i_s(0+) = 1/\Delta$, and $i_R(0+) = i_L(0+) = 0$. The current in the capacitor forces a gradual rise of the voltage across it at an initial rate of $(dv_C/dt)(0+) = i_C(0+)/C = 1/C\Delta$. Since our primary interest is in a small Δ, let us assume that during the short interval $(0,\Delta)$ the slope of the voltage curve remains constant; then the voltage reaches $1/C$ at time Δ, as shown in Fig. 2.8*b*. The current through the resistor is proportional to the voltage v_C, and hence it is linear in t (see Fig. 2.8*d*). The inductor current, being proportional to the integral of v_L, is parabolic in t (see Fig. 2.8*e*). The current through the capacitor remains constant during the interval, as shown in Fig. 2.8*c*. Of course, the assumption that during the whole interval $(0,\Delta)$ all the current from the source goes through the capacitor is false; however, the error consists of higher-order terms in Δ. Therefore, as $\Delta \to 0$ the error becomes zero. Going back to Fig. 2.8*a*, we see that as $\Delta \to 0$, i_s becomes an impulse δ, v_C undergoes a jump from 0 to $1/C$, i_C becomes an impulse δ, i_R undergoes a jump from 0 to $1/RC$, and i_L is such that $i_L(0-) = i_L(0+) = (di_L/dt)(0-) = 0$ and $(di_L/dt)(0+) = 1/LC$. Finally, as $\Delta \to 0$, from KCL we see that

$$i_C(0+) = -i_R(0+) - i_L(0+) = \frac{-1}{RC}$$

Note that these conditions check with those found earlier by other methods, as in (2.31).

3 The State-space Approach

The analysis carried out in the previous sections was a straightforward extension of the method used for first-order circuits; that is, pick one appropriate variable (i_L in the case above), and write one differential equation in this variable. Once this equation is solved, the remaining variables are easily calculated. However, there is another way of looking at the problem. It is clear that the zero-input response is completely determined once the initial conditions of the inductor current I_0 and of the capacitor voltage V_0 are known. Thus, we are led to think of I_0 and V_0 as specifying the *initial state* of the circuit; and the present state $(i_L(t),v_C(t))$ can be expressed in terms of the initial state (I_0,V_0). In other words, we may think of the behavior of the circuit as a trajectory in a two-dimensional space starting from the initial state (I_0,V_0), and for every t the corresponding point of the trajectory specifies $i_L(t)$ and $v_C(t)$.

We may legitimately ask why we need to learn this new point of view. The reason is fairly simple. First, it gives a clear pictorial description of the complete behavior of the circuit, and second, it is the only effective way to analyze nonlinear and time-varying circuits. In these more general cases, to try to select one appropriate variable and write one higher-order differential equation in terms of that variable leads to many unnecessary complications. Thus, we have a strong incentive to learn the state-space approach in the simple context of second-order linear time-invariant circuits. A further advantage is that computationally the system of equations obtained from the state-space approach is readily programmed for numerical solution on a digital computer and readily set up for solution on an analog computer. A more detailed treatment of the state-space approach will be given in Chap. 12.

3.1 State Equations and Trajectory

Consider the same parallel RLC circuit as was illustrated in Sec. 1. Let there be no current source input. We wish to compute the zero-input response. Let us use i_L and v_C as variables and rewrite Eqs. (1.1*b*) and (1.6) as follows:

$$\frac{di_L}{dt} = \frac{1}{L} v_C \qquad t \geq 0 \tag{3.1}$$

$$\frac{dv_C}{dt} = -\frac{1}{C} i_L - \frac{G}{C} v_C \qquad t \geq 0 \tag{3.2}$$

The reason that we write the equations in the above form (two simultaneous first-order differential equations) will be clear later. The variables v_C and i_L have great physical significance since they are closely related to the energy stored in the circuit. Equations (3.1) and 3.2) are first-order simultaneous differential equations and are called the *state equations* of the circuit. The pair of numbers $(i_L(t),v_C(t))$ is called the *state of the circuit at time t*. The pair $(i_L(0),v_C(0))$ is naturally called the *initial state;* it is given by the initial conditions

$$i_L(0) = I_0 \qquad v_C(0) = V_0 \tag{3.3}$$

From the theory of differential equations we know that the initial state specified by (3.3) defines uniquely, by Eqs. (3.1) and (3.2), the value of $(i_L(t),v_C(t))$ for all $t \geq 0$. Thus, if we consider $(i_L(t),v_C(t))$ as the coordinates of a point on the i_L-v_C plane, then, as t increases from 0 to ∞, the point $(i_L(t),v_C(t))$ traces a curve that starts at (I_0,V_0). The curve is called the *state-space trajectory,* and the plane (i_L,v_C) is called the *state space* for the circuit. We can think of the pair of numbers $(i_L(t),v_C(t))$ as the components of a vector $\mathbf{x}(t)$ whose origin is at the origin of the coordinate axes; thus, we write

$$\mathbf{x}(t) = \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix}$$