This looks familiar; in form it is similar to the equation resulting from the application of Kirchhoff's voltage law to the circuit of Fig. 4.1 (repeated here as Fig. 5.14b)

$$Ri + L\frac{di}{dt} = v \tag{5-27}$$

Mathematically, these two integrodifferential equations are the same and, therefore, the circuits are analogous; the solutions of the two equations are identical if one set of symbols is substituted for the other.

For exponential functions we expect a *motional impedance* $Z(s) = D + sM$. The natural behavior is defined by $Z(s) = 0$ or $s = -D/M$ and velocity $u = U_0 e^{-(D/M)t}$ decays exponentially with a time constant M/D. A steady pull $f = F$ on the mass corresponds to a "direct" force; then $Z(0) = D$, and steady velocity $U = F/D$ is the forced response to a direct force.

For a sinusoidal pull $f = \sqrt{2}\,F\cos(\omega t + \theta)$, a phasor approach is indicated where $\mathbf{F} = F/\!\underline{\theta}$. The motional impedance is

$$Z(j\omega) = \mathbf{Z} = D + j\omega M = Z/\!\underline{\phi} \tag{5-28}$$

where

$$Z = \sqrt{D^2 + (\omega M)^2} \quad \text{and} \quad \phi = \arctan\frac{\omega M}{D} \tag{5-29}$$

Then

$$\mathbf{U} = \frac{\mathbf{F}}{\mathbf{Z}} = \frac{F/\!\underline{\theta}}{Z/\!\underline{\phi}} = U/\!\underline{\theta - \phi} \tag{5-30}$$

and

$$u(t) = \sqrt{2}\,U\cos(\omega t + \theta - \phi) \tag{5-31}$$

In this case we have solved the mechanical circuit by using the techniques developed for electrical circuits. Recognizing that the two circuits are analogous, we could have gone directly to the solution by making appropriate substitutions in Eq. 5-24. The analogous terms are identified by a comparison of Eqs. 5-26 and 5-27; the results are tabulated in the first two rows of Table 5-1.

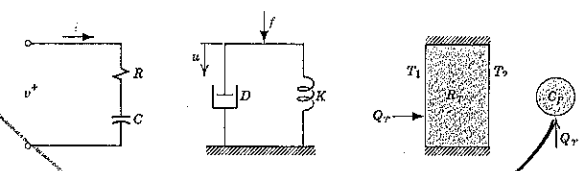
Table 5-1 Analogous Quantities

Mechanical	M	D	K	f	u	x
Electrical	L	R	C	v	i	q
Rotational	J	D_r	K_r	τ	ω	θ
Thermal		R_T	C_p	τ	q_T	w_T
Electrical II	C	G	L	i	v	λ†

† The term λ (lambda) is called flux linkage and is equal to $\int v\,dt$. (See Eq. 21-1.)

The mechanical analog for capacitance C can be obtained by comparing the governing equations for the circuits in Fig. 5.15a and b. Here friction coefficient D (analogous to R) is introduced by the *dashpot* and compliance K is introduced by the spring. From the equations

$$v = Ri + \frac{1}{C}\int i\,dt \quad \text{and} \quad f = Du + \frac{1}{K}\int u\,dt \tag{5-32}$$



(a) Electrical (b) Mechanical (c) Thermal

Figure 5.15 Other examples of analogous circuits.

it is seen that compliance K is the analog of capacitance C. Recalling that displacement $x = \int u\,dt$ and charge $q = \int i\,dt$, we see that x and q are analogous.

Two important equations in heat transfer by conduction are

$$q_T = \frac{\tau}{R_T} \quad \text{and} \quad q_T = C_p\frac{d\tau}{dt} = \frac{dw_T}{dt} \tag{5-33}$$

where q_T = rate of heat flow in joules per second (Fig. 5.15c)
R_T = thermal resistance in seconds · kelvins per joule
τ = temperature difference = $T_1 - T_2$ in kelvins
C_p = thermal capacitance in joules per kelvin
w_T = thermal energy in joules

The first equation expresses the fact that the rate of heat transfer by conduction is directly proportional to the temperature difference. The second provides for the heat absorbed by a body under conditions of changing temperature. Comparison of these equations with those describing electric circuit components leads to the thermal analogs shown in Table 5-1. Note that in this analog there is no term analogous to mass or inductance; heat flow does not exhibit any momentum or inertia effect.

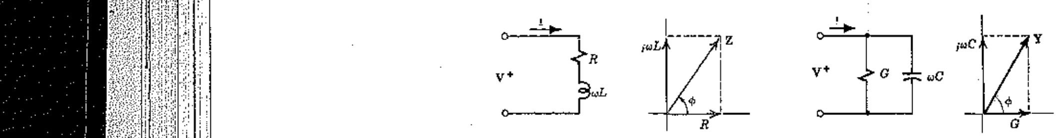
Equations 5-33 have little practical value because they imply lumped thermal effects, whereas heat conduction is always a distributed phenomenon. A further complication is that R_T and C_p vary widely with temperature. In spite of these limitations, writing the equations and identifying the analogous terms are valuable because of the insights gained when the techniques of circuit theory are applied (see Problem P5-8).

Electrical analogs have been used to study heat flow in power transistors, the production and absorption of neutrons in a nuclear reactor, and the behavior of diverse acoustical and hydraulic systems. In recognition of the ease and effectiveness of the electronic analog computer, the principles and operation of this versatile engineering tool are described in Chapter 16.

DUALS

For the series combination of R and L (Fig. 5.16a on p. 168),

$$\mathbf{Z} = R + j\omega L = Z/\!\underline{\phi} = \sqrt{R^2 + (\omega L)^2}\,/\!\underline{\arctan \omega L/R} \tag{5-34}$$



(a) R and L in series (b) G and C in parallel

Figure 5.16 Series and parallel circuits and their immittance diagrams.

For a current $\mathbf{I} = I\underline{/\theta}$,

$$\mathbf{V} = \mathbf{ZI} = Z\underline{/\phi} \cdot I\underline{/\theta} = V\underline{/\theta + \phi} \qquad (5\text{-}35)$$

The total voltage across R and L in series leads the current by phase angle ϕ, something less than 90°.

For the parallel combination of G and C (Fig. 5.16b),

$$\mathbf{Y} = G + j\omega C = Y\underline{/\phi} = \sqrt{G^2 + (\omega C)^2}\,\underline{/\arctan \omega C/G} \qquad (5\text{-}36)$$

For a voltage $\mathbf{V} = V\underline{/\theta}$,

$$\mathbf{I} = \mathbf{YV} = Y\underline{/\phi} \cdot V\underline{/\theta} = I\underline{/\theta + \phi} \qquad (5\text{-}37)$$

The total current through G and C in parallel leads the voltage by phase angle ϕ, something less than 90°.

At this time an alert student should be aware of a certain amount of repetition in the discussion of electrical circuits. The equations and diagrams for the series RL circuit of Fig. 5.16a are similar to those for the parallel GC circuit of Fig. 5.16b. With some changes in symbols, the equations and diagrams for a parallel GCL circuit (Fig. 5.9) would be just like those for the series RLC circuit of Fig. 5.8. Apparently, when we solve one electrical circuit, we automatically obtain the solution to another. How can we take advantage of this interesting fact?

In drawing up the table of mechanical–electrical analogs, we used the series RLC circuit (Fig. 5.8a). If, instead, we write the integrodifferential equation for a parallel GCL circuit similar to Fig. 5.9, we get the different set of analogs shown in the fifth row of Table 5-1. It is always possible to draw two electrical circuits analogous to a given physical system, and the corresponding terms in the two circuits are related in accordance with the principle of *duality*.

When the set of transforms that converts one system into another also converts the second into the first, the systems are said to be duals.

The series RLC circuit and the parallel GCL circuit are said to be *duals* because, by using the set of transforms indicated in Table 5-2, either circuit can be converted into the other. In general, the loop equations of a planar network have the same form as the node equations of its dual.† Some of the dual relations that exist in electrical networks are listed in Table 5-3.

A knowledge of duality enables us to double the benefit from any circuit analysis we perform. Sometimes it is advantageous to construct the dual of a given circuit instead of working with the given circuit itself. The first step (see Fig. 5.17) is to place a node in each loop of the given circuit and one more node outside; these are the nodes of the dual circuit. Then, through each element of the given circuit, draw a line terminating on the new nodes. Finally, on each line place the dual of the element through which the line is drawn; these lines are the branches of the dual circuit. The procedure is illustrated in Example 8.

† In the language of network topology, a *planar* network is one which can be drawn on a sphere with no wires that cross; every planar network has a dual.

Table 5-2 Dual Quantities

L	R	C	v	i	Z	X
C	G	L	i	v	Y	B

Table 5-3 Dual Relations

Loop current	Node voltage
Kirchhoff's voltage law	Kirchhoff's current law
Series connection	Parallel connection
Current source	Voltage source
Short circuit	Open circuit

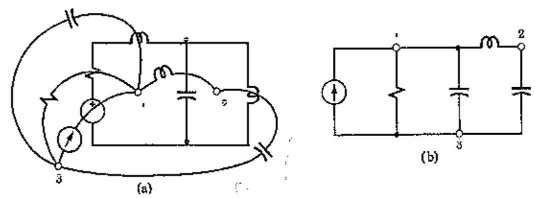


Figure 5.17 Construction of a dual circuit.

Example 8

The voltage divider of Fig. 5.18 is a useful device. Write an expression for $\mathbf{V}_2$ in terms of $\mathbf{V}$. Then draw the dual circuit and state (without derivation) the dual relation.

For the *voltage divider*,

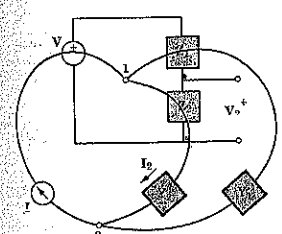
$$\frac{\mathbf{V}_2}{\mathbf{V}} = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} \qquad (5\text{-}38)$$

The *voltage* across $\mathbf{Z}_2$ is to the total *voltage* as the *impedance* $\mathbf{Z}_2$ is to the sum of the *impedances*.

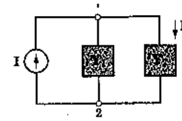
For the current divider, the dual relation is

$$\frac{\mathbf{I}_2}{\mathbf{I}} = \frac{\mathbf{Y}_2}{\mathbf{Y}_1 + \mathbf{Y}_2} \qquad (5\text{-}39)$$

The current through $\mathbf{Y}_2$ is to the total current as the admittance $\mathbf{Y}_2$ is to the sum of the admittances.



(a) Voltage divider



(b) Current divider

Figure 5.18 Deriving the dual of a circuit.

Exercise 5-16

For the circuit of Example 8, let $\mathbf{Z}_1 = R_1$ and $\mathbf{Z}_2 = R_2 + j\omega L$.

(a) Write Eqs. 5-38 and 5-39 explicitly in terms of circuit components R, L, G, and C.
(b) Identify $\mathbf{Y}_1$ and $\mathbf{Y}_2$ as specific components.

Answers: (a) $\dfrac{\mathbf{V}_2}{\mathbf{V}} = \dfrac{R_2 + j\omega L}{R_1 + R_2 + j\omega L}$, $\dfrac{\mathbf{I}_2}{\mathbf{I}} = \dfrac{G_2 + j\omega C}{G_1 + G_2 + j\omega C}$

(b) $\mathbf{Y}_1$ is a conductance $G_1 = 1/R_1$. $\mathbf{Y}_2$ is an admittance $G_2 + j\omega C$, consisting of a conductance in parallel with a capacitance.

GENERALITY OF THE IMPEDANCE APPROACH

The impedance approach is a good illustration of the power of a general method. Although the impedance concept was developed for use in determining natural response, it turns out to be applicable to forced response as well. Although it was defined in terms of exponential functions, with a proper interpretation impedance can also be used with direct and sinusoidal currents. Furthermore, when an electrical circuit has been solved, we automatically have the solution to its dual; through the analysis of an electrical circuit, we gain insight into the behavior of other analogous systems. Mastery of this important concept and the associated techniques is well worthwhile.

SUMMARY

- Impedance is defined for exponentials of the form $i = I_0 e^{st}$.
  Impedances are combined in series and parallel, just as resistances are.
  In general, the forced response is governed by $v = Z(s)i$ where
  $$Z_R(s) = R \qquad Z_L(s) = sL \qquad Z_C(s) = 1/sC$$
- For $s = 0$, $i = I_0 = I$, a direct current. In this case, $Z_R(0) = R$.
  $Z_L(0) = 0$. ∴ an inductance looks like a short circuit to a direct current.
  $Z_C(0) = \infty$. ∴ a capacitance looks like an open circuit to a direct voltage.
- A sinusoidal function of time $a = A_m \cos(\omega t + \theta)$ can be interpreted as the real part of the complex function $A_m e^{j(\omega t+\theta)} = A e^{j\theta} \cdot \sqrt{2}\, e^{j\omega t}$.
  The complex constant $Ae^{j\theta}$ is defined as phasor $\mathbf{A}$, the transform of $a(t)$.
  Phasor calculations follow the rules of complex algebra.
- Phasor voltage and current are related by the complex impedance $\mathbf{Z} = Z(j\omega)$ or the complex admittance $\mathbf{Y} = Y(j\omega)$.
  $$\mathbf{V} = \mathbf{ZI} \quad \text{where} \quad Z_R(j\omega) = R \quad Z_L(j\omega) = j\omega L \quad Z_C(j\omega) = 1/j\omega C$$
  $$\mathbf{I} = \mathbf{YV} \quad \text{where} \quad Y_R(j\omega) = G \quad Y_L(j\omega) = 1/j\omega L \quad Y_C(j\omega) = j\omega C$$
  For phasors, Kirchhoff's laws are written: $\Sigma\mathbf{V} = 0$ and $\Sigma\mathbf{I} = 0$.
- The sinusoidal response of a two-terminal network is completely defined by
  $$\mathbf{Z} = Z\underline{/\phi_Z} = R + jX \qquad \text{or} \qquad \mathbf{Y} = Y\underline{/\phi_Y} = G + jB$$

where $\phi_Z = \tan^{-1}(X/R)$ | $\phi_Y = \tan^{-1}(B/G)$
R = ac resistance in ohms | G = ac conductance in siemens
X = reactance in ohms $= \omega L$ or $-1/\omega C$ | B = susceptance in siemens $= \omega C$ or $-1/\omega L$

- To determine the forced response to sinusoids:
  1. Transform time functions to phasors and evaluate complex immittances.
  2. Combine immittances in series or parallel to simplify the circuit.
  3. Determine the desired response in phasor form.
  4. Draw a phasor diagram to check values and display results.
  5. Transform phasors to time functions if required.
- Analogous systems are described by similar integrodifferential equations.
  Corresponding terms in the equations are analog quantities.
  The solution of a problem is applicable to all analogous problems.
- Two networks are duals if the set of transforms that converts the first into the second also converts the second into the first. The loop equations of a planar network have the same form as the node equations of its dual.

TERMS AND CONCEPTS

ac impedance Ratio of phasor voltage **V** to the phasor current **I** in ohms.

admittance Ratio of current to voltage for exponential forcing function; ratio of phasor current to phasor voltage in siemens.

analogous systems Systems described by the same set of integrodifferential equations.

conductance Real part of the admittance **Y** with units of siemens.

direct current Unidirectional current of constant magnitude.

dual systems Systems such that the set of transforms that converts one system into another also converts the second system into the first.

forced response Behavior of a circuit due to an external energy source, dependent on the form of the forcing function.

phasor Transform of a sinusoidal voltage or current including the effective value and the phase angle.

phasor diagram Graphical representation of phasors and their relationship on the complex plane.

reactance Imaginary part of the impedance **Z** with units of ohms.

susceptance Imaginary part of the admittance **Y** with units of siemens.

transform Change in the mathematical description of a physical variable to facilitate computation.

REVIEW QUESTIONS

1. Define impedance.
2. Derive the dc element impedances for R, L, and C from the values of $Z(j\omega)$, letting ω approach zero as a limit.
3. In terms of steady-state response to a direct current or voltage, what is the effect of an inductance? Of a capacitance?
4. What is meant by a "long time" after closing a switch?
5. What two major advantages result from representing sinusoids by exponentials?
6. Write out in words a definition of a phasor.
7. How can a phasor, a constant quantity, represent a variable function of time?