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Bifurcation from infinity and multiplicity of solutions for nonlinear periodic boundary value problems

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Dedicated to Professor Jean Mawhin for his birthday

Abstract. *We are concerned with multiplicity and bifurcation results for solutions of nonlinear second order differential equations with general linear part and periodic boundary conditions. We impose asymptotic conditions on the nonlinearity and let the parameter vary. We then proceed to establish* a priori *estimates and prove multiplicity results (for large-norm solutions) when the parameter belongs to a (nontrivial) continuum of real numbers. Our results extend and complement those in the literature. The proofs are based on degree theory, continuation methods, and bifurcation from infinity techniques.*

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1. Introduction

We consider nonlinear second order differential equations with general linear part and periodic boundary conditions

$$\begin{gathered} u'' + b(x)u' + c(x)u + \lambda u + g(x,u) = h(x) \quad \text{a.e. in } (0,2\pi), \\ u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0, \end{gathered} \tag{1}$$

where the coefficients $b, c \in L^1(0,2\pi)$ with c bounded from above; i.e., $c(x) \leq c_0$ for a.e. $x \in (0,2\pi)$ for some (fixed) constant $c_0 \in \mathbb{R}$. The non-homogeneous term $h \in L^1(0,2\pi)$, and the nonlinearity $g : (0,2\pi) \times \mathbb{R} \to \mathbb{R}$ (which may be unbounded) is an $L^1(0,2\pi)$-Carathéodory function which is sublinear in u at infinity (i.e., $g(x,u) = \mathrm{o}(|u|)$ as $|u| \to \infty$), uniformly for a.e. $x \in (0,2\pi)$ (see conditions (C1) and (C2) below). The (real) parameter λ varies in some neighborhood of λ_1, where $\lambda_1 \in \mathbb{R}$ is the principal eigenvalue (see below) of the second order linear periodic boundary value problem

$$\begin{aligned} -u'' - b(x)u' - c(x)u = \lambda u, \quad &\text{a.e. on } (0, 2\pi), \\ u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0, \end{aligned} \tag{2}$$

where λ is a real spectral parameter.

Throughout this paper, we shall use standard notations for Lebesgue spaces $L^p(0, 2\pi)$, Sobolev spaces $W^{k,p}(0, 2\pi)$ (with $W^{k,2}(0, 2\pi)$ denoted by $H^k(0, 2\pi)$), and spaces of continuous functions $C^k([0, 2\pi])$, where k is a non-negative integer and $p \in \mathbb{R}$ with $p \geq 1$ (see e.g. [1, 6])).

It should be pointed out that all functions defined on $(0, 2\pi)$ are understood to be appropriately extended to the entire real line as 2π-periodic functions (possibly in a discontinuous fashion or in the a.e. sense if only Lebesgue measurable, for e.g., so as to agree at 0 and 2π, if need be). Also the period 2π is used only as a placeholder for convenience, any fixed period $T > 0$ will work.

By a solution to Eq.(1) we mean a function $u \in W_P^{2,1}(0, 2\pi)$ which satisfies the first equation in (1) a.e., where

$$W_P^{2,1}(0, 2\pi) := \left\{ u \in W^{2,1}(0, 2\pi) : u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0 \right\}.$$

(Observe that by the Fundamental Theorem of Calculus the space $W^{2,1}(0, 2\pi)$ is equivalent to $AC^1([0, 2\pi])$; i.e., the collection of absolutely continuous u such that u' is also absolutely continuous on $[0, 2\pi]$, see e.g. [1].)

Periodic solutions of nonlinear second order ordinary differential equations have been studied extensively. For a more recent account of the progress in this area (in the framework of resonance and nonresonance problems), we refer to the excellent monograph by A. Fonda [6]. Let us mention that when the function $g \equiv 0$, then the Fredholm Alternative type arguments describe completely the structure of the solution-set for Eq.(1) once the existence and isolation of the eigenvalue λ_1 are shown. That is, if $\lambda \neq \lambda_1$ (near λ_1), then Eq.(1) is uniquely solvable for every $h \in L^1(0, 2\pi)$. Otherwise, it is solvable only for those $h \in L^1(0, 2\pi)$ that are orthogonal (in the sense of 'duality pairing') to the eigenspace associated with λ_1, and the associated solutions can be taken as large (in an appropriate norm) as one would like since solutions are (uniquely) determined 'modulo' the associated eigenspace.

However, when $g \not\equiv 0$ is a (genuine) nonlinearity, the structure of the solution-set may be quite different from that of the linear problem. Therefore, we are interested in the solution-set structure for the nonlinear problem (1) for λ in a neighborhood of λ_1, and the nonlinearity g (which may be unbounded) satisfies some asymptotic conditions. In particular, we are concerned with the existence of multiple large-norm solutions.

Roughly speaking, in addition to a (fairly) general existence result (see Theorem 3.1), our results state that as long as the nonlinearity g satisfies (asymptotically) a 'sign-like' condition, then when λ is in an interval on *one side* of the principal eigenvalue λ_1 (see Section 2 below), Eq.(1) has at least two solutions, provided h is in an appropriate range (using the duality pairing) which includes orthogonality. Moreover, as $\lambda \to \lambda_1$ (strictly from one side), the norm of these solutions become infinitely large, whereas all solutions with λ on the other side (of λ_1) are uniformly bounded. In this way, we locate the solution-set and describe its behavior in terms of bifurcation from infinity as the parameter λ varies. Our asymptotic conditions include (very) 'strong resonances' (see Theorem 3.2); i.e., $g \to 0$ as $|u| \to \infty$ at $\lambda = \lambda_1$, and no 'decay-rate' at infinity is required; 'weaker resonances' (see Theorem 3.4) such as the so-called Landesman-Lazer type conditions (i.e., $g \nrightarrow 0$ as $|u| \to \infty$); as well as an asymptotic ('one-sided') oscillatory behavior (see Theorem 3.5); i.e., asymptotically g has infinitely many discrete-countable 'bounce-off' zeros in u. We point out that the case when the nonlinearity g is unbounded is included in our results as well.

We use an abstract set up on appropriate spaces, establish *a priori* estimates, and use a combination of degree theory (see e.g. Mawhin [10]), continuation methods and Rabinowitz bifurcation from infinity techniques ([7, 12, 17, 18, 19, 20]) to prove our results. An important ingredient in obtaining the necessary estimates is the use of comparison principles and estimates for the linear problem obtained in Section 2 below (under somewhat weaker conditions than those usually considered in the literature; particularly in the one-dimensional case (see e.g. [3, 4, 16])).

Let us recall that some results on multiplicity or bifurcation from infinity for nonlinear problems with periodic boundary conditions have been obtained before under a different set of conditions (see e.g. [5, 6, 8, 12] and references therein). However, our results are more in line with those in [12, 13] and references therein; herein we consider a more general linear part and more general nonlinearities.

We wish to mention that a systematic study of periodic solutions of (autonomous) nonlinear differential equations with small parameters was initiated by H. Poincaré in his celebrated treatise on celestial mechanics ([14]) in connection with the three body problem (also see [11, 15]). Since then, a great deal of work has been devoted to the study of periodic solutions of nonlinear differential equations depending on parameters in many different directions; especially using homotopy, continuation, as well as global methods (see e.g. [6, 7, 10, 17, 18]). In the last fifty years Professor Mawhin has tremendously contributed in an unparalleled way to the development of the theory of periodic solutions of nonlinear differential equations; which most likely served as a catalyst to his introducing the coincidence degree theory ([10]); an extension of Leray-Schauder degree to nonlinear problems which cannot necessarily be written as compact perturbations of the identity. It is with an immense gratitude that we write this paper on periodic solutions in his honor.

This paper is organized as follows. In Section 2, we consider the linear problem and obtain the necessary comparison principles and estimates that will be needed for the nonlinear problem. As indicated above, these results are of independent interest in their own right. In Section 3, we give the general assumptions on the data, state our main results for nonlinear problems, and give some simple illustrative examples (for the reader's convenience) along the way. In Section 4, we cast the problem in an abstract setting and establish the necessary *a priori* estimates for possible solutions. Finally, Section 5 is devoted to the proofs of our main results. Remarks are included throughout as appropriate, and a visual rendition sketch of a bifurcation diagram for a 'bounce-off' oscillatory nonlinearity is given in Section 3.

2. A general periodic linear eigenproblem and estimates

In this section, we consider the issue of comparison principle(s) and the existence of a (unique) principle eigenvalue for a general (i.e., not necessarily symmetric) linear periodic problems with (possibly) unbounded coefficients. We also obtain some estimates on the linear problem that will prove useful when considering nonlinear problems.

Pick $\mu \in \mathbb{R}$ be such that $\mu > c_0$; which implies that $\mu - c(x) \geq \mu - c_0 > 0$ for a.e. $x \in (0, 2\pi)$. Consider the ('augmented') linear differential operator defined on $W_P^{2,1}(0, 2\pi)$ by

$$L_\mu u = -u'' - b(x)u' - c(x)u + \mu u. \tag{3}$$

We first set $a(x) := e^{\int_0^x b(s)\, ds}$, and multiply $L_\mu u$ by the 'integrating factor' $a(x)$. It follows that the operator L_μ is transformed into the linear differential operator

$$S_\mu u := -\left(a(x)u'\right)' + a(x)(\mu - c(x))u; \tag{4}$$

which (despite its appearance) is not necessarily symmetric on $W_P^{2,1}(0, 2\pi)$.

Observe that a pair (λ, φ) with $\varphi \in W_P^{2,1}(0, 2\pi) \setminus \{0\}$ is an eigenpair for the eigenvalue problem

$$L_\mu u = \lambda u \tag{5}$$

if and only if it is also an eigenpair for the eigenvalue problem with weight

$$S_\mu u = \lambda a(x)u. \tag{6}$$

We shall show that the eigenvalue problem (5) has a (real) positive principal eigenvalue with a positive (i.e., bounded away from zero) eigenfunction on the closed interval $[0, 2\pi]$, even when $b, c \in L^1(0, 2\pi)$ are not necessarily locally bounded (with c bounded from above only), as indicated. We first investigate some properties of the linear differential operator L_μ on the space $W^{2,1}_P(0,2\pi)$. As a first result in that direction, we have the following order-preserving or weak minimum/comparison principle.

Proposition 2.1. *Suppose that $u \in W^{2,1}_P(0,2\pi)$ satisfies the differential inequality $L_\mu u \geq 0$ for a.e. $x \in (0,2\pi)$. Then $u \geq 0$ on $[0,2\pi]$.*

Proof. Let $u \in W^{2,1}_P(0,2\pi)$ be such that $L_\mu u \geq 0$ for a.e. $x \in (0,2\pi)$, by using the 'integrating factor' $a(x) := e^{\int_0^x b(s)\,ds}$, it follows immediately that $S_\mu u \geq 0$ for a.e. $x \in (0,2\pi)$; which implies that $(a(x)u')' \leq a(x)(\mu - c(x))u$ for a.e. $x \in (0,2\pi)$.

Now, suppose that $u(x) < 0$ for some $x \in [0,2\pi]$, then u has a negative minimum value in this interval, say at $x_0 \in [0,2\pi]$. Therefore, there is a neighborhood $I_\delta := (x_0 - \delta, x_0 + \delta)$ such that $u(x_0) \leq u(x) < 0$ for all $x \in I_\delta$ and $u'(x_0) = 0$, where we have used the continuity of $u(x)$ and (possibly) the 2π-periodic extension of u if x_0 is an end-point of the interval $[0,2\pi]$. It follows that $(a(x)u')' \leq a(x)(\mu - c(x))u < a(x)(\mu - c_0)u < 0$ for a.e. $x \in I_\delta$. The Fundamental Theorem of Calculus immediately implies that $a(x)u'$ is (strictly) decreasing in I_δ. Since $u'(x_0) = 0$ (i.e., $a(x_0)u'(x_0) = 0$), we obtain that $a(x)u'(x) > 0$ for $x \in (x_0 - \delta, x_0)$ and $a(x)u'(x) < 0$ for $x \in (x_0, x_0 + \delta)$; that is, $u'(x) > 0$ for $x \in (x_0 - \delta, x_0)$; which implies that $u(x)$ is (strictly) increasing in $(x_0 - \delta, x_0)$. This is a contradiction with the fact that $u(x_0)$ is a (negative) minimum value of the function u. Therefore, $u(x) \geq 0$ on $[0,2\pi]$, and the proof is complete. $\square$

This proposition immediately implies that $\lambda = 0$ is not an eigenvalue of the differential operator L_μ in Eq.(5), since any possible eigenfunction would be identically zero in this case. We now want to show that $\lambda = 0$ is actually in the 'resolvent' of L_μ; that is; to show that the equation $L_\mu u = e(x)$ has a (unique) solution $u \in W^{2,1}_P(0,2\pi)$ for every $e \in L^1(0,2\pi)$. For that purpose, we need the following *a priori* estimate; which will be also useful in studying nonlinear problems.

Lemma 2.2. *There exists a constant $\alpha := \alpha(b,c,\mu) > 0$ such that*

$$|L_\mu u|_{L^1(0,2\pi)} \geq \alpha |u|_{W^{2,1}_P(0,2\pi)} \quad \textit{for all } u \in W^{2,1}_P(0,2\pi). \tag{7}$$

Proof. Suppose the conclusion doe not hold. Then, there is a sequence $(u_n) \subset W^{2,1}_P(0,2\pi) \setminus \{0\}$ such that for all $n \in \mathbb{N}$ one has that

$$|L_\mu u_n|_{L^1(0,2\pi)} \leq \frac{1}{n} |u_n|_{W^{2,1}_P(0,2\pi)}.$$

Setting $v_n := u_n / |u_n|_{W^{2,1}_P(0,2\pi)}$ and $L_\mu v_n = h_n$, we get that $|v_n|_{W^{2,1}_P(0,2\pi)} = 1$ for all $n \in \mathbb{N}$, and that $h_n \to 0$ in $L^1(0,2\pi)$ as $n \to \infty$. By the continuous imbedding of $W^{2,1}_P(0,2\pi)$ into $C^1_P[0,2\pi]$, one has that there exist a constant $C_1 > 0$

(independent of n) such that $|v_n|_{C^1_P[0,2\pi]} \leq C_1$. Moreover, since $W^{2,1}_P(0,2\pi)$ is compactly imbedded into $W^{1,1}_P(0,2\pi)$, one has (by going to a subsequence relabeled (v_n), if need be) that there is a function $v \in W^{1,1}_P(0,2\pi)$ such that $v_n \to v$ in $W^{1,1}_P(0,2\pi)$ as $n \to \infty$; which implies (for a subsequence similarly relabeled if need be) that $v_n(x) \to v(x)$ and $v'_n(x) \to v'(x)$ for a.e. $x \in (0,2\pi)$ (see e.g. [1, Theorem 4.9]). Since $|b(x)v'_n(x)| \leq C_1|b(x)|$ and $|c(x)v_n(x)| \leq C_1|c(x)|$ for a.e. $x \in (0,2\pi)$, it follows from the Lebesgue Dominated Convergence Theorem that $b(\cdot)v'_n \to b(\cdot)v'$ and $c(\cdot)v \to c(\cdot)v$ in $L^1(0,2\pi)$ as $n \to \infty$. This and the fact that $v''_n = -h_n - b(x)v'_n - c(x)v_n + \mu v_n$ imply that $v''_n \to -b(x)v' - c(x)v + \mu v$ in $L^1(0,2\pi)$ with $v_n \to v$ in $W^{1,1}_P(0,2\pi)$ as $n \to \infty$. The (strong) closedness (see e.g. [1, p. 204, Remark4]) of the differentiation-operator from $W^{1,1}_P(0,2\pi)$ into $L^1(0,2\pi)$ implies that $v \in W^{2,1}_P(0,2\pi)$ and that $v_n \to v$ in $W^{2,1}_P(0,2\pi)$ as $n \to \infty$ with $v'' = -b(x)v' - c(x)v + \mu v$ for a.e. $x \in (0,2\pi)$; that is, $L_\mu v = 0$ for a.e. $x \in (0,2\pi)$. It follows immediately from Proposition 2.1 that $v \equiv 0$. This is a contradiction with the fact that $|v_n|_{W^{2,1}_P(0,2\pi)} = 1$ for all $n \in \mathbb{N}$ and $v_n \to v$ in $W^{2,1}_P(0,2\pi)$ as $n \to \infty$. The proof is complete. $\square$

Since the linear operator $L_\mu : W^{2,1}_P \Subset L^1(0,2\pi) \to L^1(0,2\pi)$ is compactly and densely defined, takes bounded sets in $W^{2,1}_P(0,2\pi)$ into bounded sets in $L^1(0,2\pi)$ and is one-to-one (see Lemma 2.2), we claim that it is onto $L^1(0,2\pi)$; i.e., L_μ is invertible on $L^1(0,2\pi)$. In fact, one has the following existence (and uniqueness) result.

LEMMA 2.3. *For every $e \in L^1(0,2\pi)$, the equation $L_\mu u = e(x)$ a.e. in $(0,2\pi)$ has a (unique) 2π-periodic solution $u \in W^{2,1}_P(0,2\pi)$.*

Proof. Uniqueness follows from Proposition 2.1 or Lemma 2.2. To prove existence, we use the topological degree theory by considering the homotopy

$$-u'' + \theta\left(-b(x)u' + (\mu - c(x))u\right) + (1-\theta)(\mu - c_0)u = \theta e(x) \quad \text{a.e. in } (0,2\pi),$$

where $\theta \in [0,1]$. Notice that the homotopy reduces to the equation $L_\mu u = e(x)$ when $\theta = 1$, and when $\theta = 0$ it reduces to the periodic linear differential equation with constant coefficients $-u'' + (\mu - c_0)u = 0$ on $[0,2\pi]$, where $\mu - c_0 > 0$. It therefore suffices to show that all possible solutions to the homotopy are (uniformly) bounded in $W^{2,1}_P(0,2\pi)$ independently of $\theta \in [0,1]$. Indeed, suppose that this is not the case, then one can find sequences $(u_n) \subset W^{2,1}_P(0,2\pi) \setminus \{0\}$ and $(\theta_n) \subset [0,1]$ such that for all $n \in \mathbb{N}$, $|u_n|_{W^{2,1}_P(0,2\pi)} \geq n$ and

$$u''_n = \theta_n\left(-b(x)u'_n + (\mu - c(x))u_n\right) + (1-\theta_n)(\mu - c_0)u_n - \theta_n e(x) \text{ a.e. in } (0,2\pi).$$

Setting $v_n := u_n/|u_n|_{W^{2,1}_P(0,2\pi)}$ and using the fact that $W^{2,1}_P(0,2\pi)$ is continuously imbedded into $C^1_P[0,2\pi]$ and compactly imbedded into $W^{1,1}_P(0,2\pi)$, the

Lebesgue Dominated Convergence Theorem, the closedness of the differentiation operator, and arguments similar to those used in the proof of Lemma 2.2, it follows that there exist $v \in W_P^{2,1}(0, 2\pi)$ and $\theta_0 \in [0, 1]$ such that (by going if necessary to subsequences similarly relabeled) $v_n \to v$ in $W_P^{2,1}(0, 2\pi)$, $\theta_n \to \theta_0$ as $n \to \infty$, and v satisfies the homogeneous linear equation

$$-v'' - \theta_0 b(x)v' + \theta_0(\mu - c(x))v + (1 - \theta_0)(\mu - c_0)v = 0 \quad \text{for a.e. in } (0, 2\pi).$$

Since $\theta_0 b \in L^1(0, 2\pi)$ and $\theta_0(\mu - c(x)) + (1 - \theta_0)(\mu - c_0) \geq \mu - c_0 > 0$ for a.e. $x \in (0, 2\pi)$, it follows from arguments used in the proof of Proposition 2.1 that $v \geq 0$ and $v \leq 0$; that is, $v = 0$. This is a contradiction with the fact that $|v_n|_{W_P^{2,1}(0,2\pi)} = 1$ for all $n \in \mathbb{N}$ and $v_n \to v$ in $W_P^{2,1}(0, 2\pi)$ as $n \to \infty$. The proof is complete. □

Now, we wish to show that a strong minimum/comparison principle also holds for the differential operator L_μ under the weak assumptions imposed on the coefficient-functions b and c. That is, a strong positivity or strong order preserving property holds for the second order differential operator L_μ. (Some techniques from [21] and periodicity prove useful here.)

Proposition 2.4. *Suppose that $u \in W_P^{2,1}(0, 2\pi)$ satisfies the differential inequality $L_\mu u \geq 0$ for a.e. $x \in (0, 2\pi)$ with $u \not\equiv 0$, then $u > 0$ on the closed interval $[0, 2\pi]$; that is u is positive and bounded away from zero on the whole closed interval $[0, 2\pi]$, unless it is identically zero.*

Proof. Since $u \in W_P^{2,1}(0, 2\pi)$ satisfies the differential inequality $L_\mu u \geq 0$ for a.e. $x \in (0, 2\pi)$, one has immediately that $S_\mu u \geq 0$ for a.e. $x \in (0, 2\pi)$. Moreover, it follows from Proposition 2.1 that $u(x) \geq 0$ for all $x \in [0, 2\pi]$. Since $u \not\equiv 0$ is 2π-periodic, one has that either $u > 0$ on $[0, 2\pi]$ (in which case the conclusion holds), or otherwise, one may assume (without loss of generality) that there is a point $x_0 \in (0, 2\pi]$ such that $u(x_0) = 0$ and $u(x) > 0$ for all $x \in (x_0 - \delta, x_0)$, where $\delta \in (0, 2\pi)$ is a (fixed) constant; that is, the function u has a strict local minimum at a point x_0 in a (deleted) left-neighborhood of x_0; which implies that $u'(x_0) \leq 0$. Actually, the 2π-periodicity of u implies that $u'(x_0) = 0$ for otherwise one reaches a contradiction in the light of Proposition 2.1 (by possibly extending the function u periodically if $x_0 = 2\pi$, and hence $x_0 = 0$ as well). It follows that $a(x_0)u'(x_0) = 0$, and by using the Fundamental Theorem of Calculus and (4), one has that $u(x) = \int_{x_0}^{x} u'(s)\,ds$ and that $-a(x)u'(x) \leq \int_x^{x_0} a(s)(\mu - c(s))u(s)\,ds$ for all $x \in (x_0 - \delta, x_0)$. This implies that $-u'(x) \leq v(x)\left(a_0 \int_0^{2\pi} a(s)(\mu - c(s))\,ds\right)$, where $v(x) := \max\limits_{s \in [x, x_0]} u(s) > 0$ and $a_0^{-1} = \min\limits_{s \in [0, 2\pi]} a(s)$. Therefore, by the Fundamental Theorem of Calculus again, one has that

$$u(x) \le (x_0 - x)v(x)\left(a_0 \int_0^{2\pi} a(s)(\mu - c(s))\,ds\right) \text{ for all } x \in (x_0-\delta, x_0).$$

For every $n \in \mathbb{N}$ such that $n > 1/\delta$, let $x_n \in [x_0 - \frac{1}{n}, x_0)$ be a point such that $\max_{[x_0-\frac{1}{n},x_0]} u(s) := u(x_n)$; which exists since the function u is continuous on the compact interval $[x_0 - \frac{1}{n}, x_0]$. Given that $[x_n, x_0] \subset [x_0 - \frac{1}{n}, x_0]$, it follows that $v(x_n) := \max_{[x_n,x_0]} u(s) = u(x_n)$, and $0 < x_0 - x_n \le 1/n$ for all $n \in \mathbb{N}$ such that $n > 1/\delta$. Therefore, by setting $A := a_0 \int_0^{2\pi} a(s)(\mu - c(s))\,ds$, one has that

$$0 < u(x_n) \le (x_0 - x_n)v(x_n)A \le \frac{A}{n}v(x_n) = \frac{A}{n}u(x_n) < u(x_n)$$

for all $n \in \mathbb{N}$ such that $n > \max(A, 1/\delta)$. This is a contradiction. Thus, $u(x) > 0$ on the closed interval $[0, 2\pi]$, and the proof is complete. $\square$

Now, we let $K := \{u \in H_P^1(0,2\pi) : u \ge 0\} \subset H_P^1(0,2\pi)$ be the (solid) cone with non-empty interior. Setting $T_\mu := L_\mu^{-1} : L^1(0,2\pi) \to W_P^{2,1}(0,2\pi) \Subset L^1(0,2\pi)$, it follows from Lemma 2.3 that (the scalar) zero is not an eigenvalue of the compact linear operator $T_\mu : L^1(0,2\pi) \to L^1(0,2\pi)$; although, it is always in the spectrum of T_μ (see e.g. [1, p. 164, Theorem 6.8]). Moreover, due to Proposition 2.4, one can show that T_μ has a positive spectral radius $r := r(T_\mu) > 0$. By Proposition 2.1, one has that $T_\mu(K) \subset K$. Since (the restriction) $T_\mu : H_P^1(0,2\pi) \to H_P^1(0,2\pi)$ satisfies all the assumptions of the Krein-Rutman Theorem, it follows that $r(T_\mu)$ is a (real) positive eigenvalue of T_μ with an eigenfunction $\phi_1 \in K$, $\phi_1 \not\equiv 0$. In addition, $r(T_\mu^*) = r(T_\mu)$ is also an eigenvalue of the adjoint T_μ^* with an eigenfunction $\phi_1^* \in K^* := \left\{f \in \left(H_P^1(0,2\pi)\right)^* : f(x) \ge 0 \text{ for all } x \in K\right\}$ called the dual cone of K; which in this instance is also a cone in $\left(H_P^1(0,2\pi)\right)^*$ since one can easily show that $K^* \cap (-K^*) = \{0\}$ by using the definition of K^* and the fact that $H_P^1(0,2\pi) = K - K$ (i.e., the cone K "reproduces" the space $H_P^1(0,2\pi)$).

Before proceeding, we want to make a few observations that will be needed later on. First observe that $\phi_1 \in W_P^{2,1}(0,2\pi)$ since it is in the range of T_μ (i.e., regularity of solutions). Also, notice that by using the (equivalent) inner product $(u,v) := \int_0^{2\pi} u'v'\,dx + \int_0^{2\pi}(\mu - c(x))uv\,dx$ for all $u, v \in H_P^1(0,2\pi)$ (or simply the standard inner product), it follows from the Riesz-Fréchet Representation Theorem (see e.g. [1, p. 135, Theorem 5.5]) that the Hilbert space $H_P^1(0,2\pi)$ may be (isometrically) identified with its dual; i.e., $\left(H_P^1(0,2\pi)\right)^* \cong H_P^1(0,2\pi)$), and hence ϕ_1^* may be identified with an element of $H_P^1(0,2\pi)$, still denoted by $\phi_1^* \in H_P^1(0,2\pi) \subset L^\infty(0,2\pi)$. Furthermore, using the fact that the dual $\left(L^1(0,2\pi)\right)^* = L^\infty(0,2\pi)$ by the Riesz Representation Theorem (see e.g. [1, p. 99, Theorem 4.11]), one has that the duality pairing $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{(L^\infty(0,2\pi)),L^1(0,2\pi))}$ implies that

$$\langle L_\mu^*(\phi_1^*), u\rangle \stackrel{\text{def}}{=} \langle \phi_1^*, L_\mu(u)\rangle_{(L^\infty(0,2\pi),L^1(0,2\pi))} = (\phi_1^*, u) - \int_0^{2\pi} b(x)\phi_1^* u'\,dx$$

for all $u \in \mathrm{Dom}(L_\mu) = W_P^{2,1}(0,2\pi) \subset L^1(0,2\pi)$ (see e.g. [1, p. 44]); that is,

$$\begin{aligned}\langle L_\mu^*(\phi_1^*), u\rangle = \langle \phi_1^*, L_\mu(u)\rangle = \int_0^{2\pi} \phi_1^{*\prime} u'\,dx + &\int_0^{2\pi} (\mu - c(x))\phi_1^* u\,dx \\ - &\int_0^{2\pi} b(x)\phi_1^* u'\,dx\end{aligned}$$

for all $u \in \mathrm{Dom}(L_\mu) = W_P^{2,1}(0,2\pi)$. This type of identity holds true for L_0 and L_0^* as well (i.e., when $\mu = 0$); it boils down to multiplying $\phi_1^* \in H_P^1(0,2\pi) \subset L^\infty(0,2\pi)$ by $L_0(u)$ for any $u \in W_P^{2,1}(0,2\pi)$ and integrating over $[0,2\pi]$. (It will be used repeatedly in the sequel.)

Now, under the weaker conditions imposed on the coefficients of the linear operator L_μ, it follows from Proposition 2.4 above and the (stronger version of) the Krein-Rutman Theorem that ϕ_1 is in the interior of the cone K and that the corresponding eigenvalue is simple. However, by using the periodicity of ϕ_1 and the uniqueness of solutions to linear initial value problems, we present below a shorter and simpler proof adapted to our specific situation since it also allows us to get more information on the ('dual') eigenfunction ϕ_1^*. Indeed, we have the following result.

Proposition 2.5. *The linear spectral problem*

$$L_0 u := -u'' - b(x)u' - c(x)u = \lambda u, \quad u \in W_P^{2,1}(0,2\pi), \tag{8}$$

has a real simple eigenvalue λ_1 with nonnegative eigenfunction $\phi_1 \in W_P^{2,1}(0,2\pi)$ which is actually positive; i.e., bounded away from zero on the whole closed interval $[0,2\pi]$. Moreover, λ_1 is also a real eigenvalue of the adjoint operator L_0^* of L_0 with a nonnegative eigenfunction ϕ_1^*.

If, in addition, the coefficient $b \in AC_P([0,2\pi]) = W_P^{1,1}(0,2\pi)$, then ϕ_1^* is also positive on the closed interval $[0,2\pi]$.

Proof. As above, we first consider the ('augmented') invertible linear operator L_μ given by $L_\mu u = -u'' - b(x)u' + (\mu - c(x))u$ whose inverse is denoted by T_μ. Then, by the Krein-Rutman Theorem, the spectral problem $T_\mu\phi = \lambda\phi$ has a (real) eigenvalue $\lambda := r(T_\mu) > 0$ with a nonnegative eigenfunction ϕ_1 as indicated above. Applying L_μ on both sides, one deduces that $L_\mu\phi_1 = (r(T_\mu))^{-1}\phi_1$;

which implies immediately that $\lambda_1 := (r(T_\mu))^{-1} - \mu$ is an eigenvalue of the operator $L_0 u := -u'' - b(x)u' - c(x)u$ with nonnegative eigenfunction ϕ_1, and that it is also an eigenvalue of the operator L_0^* with a nonnegative eigenfunction ϕ_1^*. Now, if there is $x_0 \in [0, 2\pi]$ such that $\phi_1(x_0) = 0$, then x_0 is a minimum point for ϕ_1, and hence (extending ϕ_1 by 2π-periodicity if x_0 is a boundary point) $\phi_1'(x_0) = 0$ as well since $\phi_1 \in W_P^{2,1}(0, 2\pi) \subset C_P^1([0, 2\pi])$. Therefore, uniqueness results for (Carathéodory) solutions (see e.g. [21]) to initial value problems for second order homogeneous linear ordinary differential equations with $L^1(0, 2\pi)$-coefficients (written as integral solutions to a first order system and use of generalized Gronwall's inequality on their norm) would imply that the only solution to $L_0\phi_1 - \lambda_1\phi_1 = 0$ a.e. is given by $\phi_1 \equiv 0$ on $[0, 2\pi]$; which would contradict the fact that ϕ_1 is an eigenfunction. Thus ϕ_1 is positive (and hence bounded away from zero) on $[0, 2\pi]$ as needed.

To show that λ_1 is simple, let $w \in W_P^{2,1}(0, 2\pi)$ be an eigenfunction associated with λ_1. Then, one has that $L_0(\phi_1 + tw) = \lambda_1(\phi_1 + tw)$ for all $t \in \mathbb{R}$. Since ϕ_1 is positive on $[0, 2\pi]$ and w is continuous, it follows that for $|t|$ small one has that $\phi_1 + tw$ remains positive on $[0, 2\pi]$, and that for some $t \in \mathbb{R}$ with $|t|$ large, $\phi_1 + tw$ does not remain positive on $[0, 2\pi]$ since $w \not\equiv 0$. Therefore, by continuity (and connectedness), one has that there is $t_0 \in \mathbb{R}$ such that $(\phi_1 + t_0 w)(x) \geq 0$ on $[0, 2\pi]$, and $(\phi_1 + t_0 w)(x_0) = 0$ for some $x_0 \in [0, 2\pi]$ with $L_0(\phi_1 + t_0 w) - \lambda_1(\phi_1 + t_0 w) = 0$ a.e. on $(0, 2\pi)$. The above uniqueness argument implies that $(\phi_1 + t_0 w) \equiv 0$ on $[0, 2\pi]$; that is, $w = -t_0^{-1}\phi_1$, and the simplicity of λ_1 follows.

If in addition $b \in AC_P([0, 2\pi]) = W_P^{1,1}(0, 2\pi)$, then one has that $(b\phi_1^*) \in W_P^{1,1}(0, 2\pi)$. Using integration by parts in the pairing, one has that $\phi_1^* \in H_P^1(0, 2\pi)$ satisfies

$$\int_0^{2\pi} \phi_1^{*\prime} u'\, dx = -\int_0^{2\pi} (b\phi_1^*)' u\, dx + \int_0^{2\pi} c(x)\phi_1^* u\, dx + \lambda_1 \int_0^{2\pi} \phi_1^* u\, dx$$

for every $u \in \mathrm{Dom}(L_0) = W_P^{2,1}(0, 2\pi)$, and hence in particular for every $u \in C_0^\infty(0, 2\pi)$; which implies that $\phi_1^* \in W^{2,1}(0, 2\pi)$ by the definition of the Sobolev space $W^{1,1}(0, 2\pi)$ (see e.g. [1, p. 202]). Since $(bv)' = b'v + bv' \in L^1(0, 2\pi)$ for every $v \in W^{1,1}(0, 2\pi) = AC([0, 2\pi])$, and the (formal) adjoint linear operator L_0^* is explicitly given by

$$L_0^* v = -v'' + (b(x)v)' - c(x)v = -v'' + b(x)v' - (c(x) - b'(x))v,$$

it follows that $L_0^*(\phi_1^*) - \lambda_1\phi_1^* = 0$ a.e. on $(0, 2\pi)$ with $\phi_1^* \in W_P^{2,1}(0, 2\pi)$. The nonnegativity of ϕ_1^* and the above uniqueness arguments can now be used to show that ϕ_1^* is positive on the closed interval $[0, 2\pi]$. The proof is complete. □

The following result will prove useful in obtaining *a priori* estimates for possible solutions to some nonlinear periodic problems in subsequent sections.

Proposition 2.6. *There exists a constant $\lambda_0 > 0$ such that for all $p \in L^1(0, 2\pi)$ with $0 \le p(x) \le \lambda_0$ and all $u \in W_P^{2,1}(0, 2\pi)$ satisfying a.e. the equation*

$$u'' + b(x)u' + c(x)u + \lambda_1 u + p(x)u = 0,$$

one has that either $u = 0$ on $[0, 2\pi]$ or $\min_{[0,2\pi]} |u(x)| > 0$ (i.e., u is either positive or negative on $[0, 2\pi]$).

Proof. Since $u \equiv 0$ is a solution to the (homogeneous linear periodic) equation, we may suppose without loss of generality that $u \in W_P^{2,1}(0, 2\pi) \setminus \{0\}$, and we claim that under the above assumptions one must have that $\min_{[0,2\pi]} |u(x)| > 0$. Indeed, assume that the conclusion of the proposition does not hold. Then, for every $n \in \mathbb{N}$ there exist $p_n \in L^1(0, 2\pi)$ with $0 \le p_n(x) \le 1/n$ a.e. and $u_n \in W_P^{2,1}(0, 2\pi)$ with $|u_n|_{W^{2,1}(0,2\pi)} = 1$ such that $\min_{[0,2\pi]} |u_n(x)| = 0$ and for a.e. $x \in (0, 2\pi)$ one has that

$$u_n'' + b(x)u_n' + c(x)u_n + \lambda_1 u_n + p_n(x)u_n = 0.$$

Using the fact that $W_P^{2,1}(0, 2\pi)$ is continuously imbedded into $C_P^1[0, 2\pi]$ and compactly imbedded into $W_P^{1,1}(0, 2\pi)$, the Lebesgue Dominated Convergence Theorem, the closedness of the differentiation operator, and arguments similar to those used in the proof of Lemma 2.2, it follows (by going if necessary to subsequence relabeled (u_n)) that there exist $u \in W_P^{2,1}(0, 2\pi) \setminus \{0\}$ such that $u_n \to u$ in $W^{2,1}(0, 2\pi)$, $|u|_{W^{2,1}(0,2\pi)} = 1$, and $u'' + b(x)u' + c(x)u + \lambda_1 u = 0$. Therefore, u is an eigenfunction associated with the simple eigenvalue λ_1, and hence is proportional to ϕ_1. Thus, it has one sign and is bounded away from zero by Proposition 2.5; i.e., $\min_{[0,2\pi]} |u(x)| > 0$. This fact and the uniform convergence of u_n to u in $C_P^0[0, 2\pi]$ imply that there is $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ one has that $\min_{[0,2\pi]} |u_n(x)| > 0$. This is a contradiction, and the proof is complete. □

Remark 2.7. Propositions 2.4 and 2.5 may be used (in conjunction with the Krein-Rutman Theorem) to show that the eigenvalue λ_1 is principal and unique; i.e., it is the only (real) eigenvalue with a positive eigenfunction ϕ_1 and one-dimensional eigenspace (see e.g. [1]). Moreover, an analysis of the proof of Proposition 2.1 and the result in Proposition 2.5 show that if $\lambda \ne \lambda_1$ is a real eigenvalue of the spectral problem (8), then $\lambda > \lambda_1$. Indeed, if $\lambda < \lambda_1$ is an eigenvalue of Eq.(8) with eigenfunction $u \in W_P^{2,1}(0, 2\pi)$; i.e., $L_0 u + \lambda u = 0$ a.e. on $[0, 2\pi]$, then using the fact that ϕ_1 is positive on $[0, 2\pi]$ and setting $v := u/\phi_1$, one has that $-\lambda v = [\phi_1(\cdot)]^{-1} L_0(v\phi_1)$. Using direct calculations of $L_0(v\phi_1)$ through the product rule for derivatives and collecting terms, it follows easily that $v \in W_P^{2,1}(0, 2\pi)$ satisfies a.e. the homogeneous linear differential equation

$v'' + d(x)v' + (\lambda - \lambda_1)v = 0$ with $\lambda - \lambda_1 < 0$, where $d(x) = b(x) + 2\phi_1'(x)/\phi_1(x)$. Now, arguments similar to those used in the proof of Proposition 2.1 imply that $v \geq 0$ and $-v \geq 0$ for all $x \in [0, 2\pi]$. That is, $v \equiv 0$, and hence $u \equiv 0$; contradicting the fact that $u \neq 0$ is an eigenfunction.

3. Main results

From now on, we shall write the nonlinear equation (1) in the equivalent form

$$\begin{gathered} u'' + b(x)u' + c(x)u + \lambda_1 u + \lambda u + g(x,u) = h(x) \quad \text{a.e. in } (0,2\pi), \\ u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0, \end{gathered} \tag{9}$$

where $\lambda_1 \in \mathbb{R}$ is the principal eigenvalue obtained in Proposition 2.5, and the parameter $\lambda \in \mathbb{R}$ will vary in a neighborhood of zero. Therefore, Eq.(1) is equivalent to

$$\begin{gathered} Lu + \lambda u + g(x,u) = h(x) \quad \text{a.e. in } (0,2\pi), \\ u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0, \end{gathered} \tag{10}$$

where the linear operator $L : W_P^{2,1}(0,2\pi) \to L^1(0,2\pi)$ is defined by

$$Lu := u'' + b(x)u' + c(x)u + \lambda_1 u$$

for which the scalar $\lambda = 0$ is the principal eigenvalue with associated (positive) eigenfunction ϕ_1. (Notice that $\lambda = 0$ is also a principal eigenvalue of the adjoint L^* of L with associated nonnegative eigenfunction $\phi_1^* \neq 0$.)

In this section we state our general assumptions on the nonlinearity g and the function h. We assume that $g : (0,2\pi) \times \mathbb{R} \to \mathbb{R}$ is an $L^1(0,2\pi)$-Carathéodory function which is sublinear at infinity in u, uniformly a.e. in x, and satisfies 'sign-like' conditions. We also impose asymptotic conditions on g and their relationship with the forcing term h. These conditions include, among others, strong resonance conditions, Landesman-Lazer type conditions, as well as oscillatory conditions. (Some results herein were motivated by [9].)

In addition to a (fairly) general existence result, we state our main results on multiplicity of solutions (with large norms for λ 'small') when λ is in an interval *on one side* of the first eigenvalue, and the existence of (at least) one solution for λ on the other side. The existence of a third solution (with a somewhat 'smaller norm') is also discussed. Simple examples are provided to motivate and illustrate the results.

As mentioned above, we specifically assume the following general conditions; the first three of which refer to the nonlinearity g, whereas the last one relate the nonhomogeneous term h to the asymptotic behavior of g and the null-space associated with the eigenvalue λ_1.

(C1) $g(\cdot, u)$ is measurable for all $u \in \mathbb{R}$, $g(x, \cdot)$ is continuous for a.e. $x \in (0, 2\pi)$, and for every $r > 0$ there is a function $\gamma_r \in L^1(0, 2\pi)$ such that

$$|g(x, u)| \leq \gamma_r(x), \tag{11}$$

for a.e. $x \in (0, 2\pi)$ and all $u \in \mathbb{R}$ with $|u| \leq r$.

(C2) $\lim\limits_{|u|\to\infty} \dfrac{g(x, u)}{u} = 0$ uniformly a.e. in x; that is, for every $\varepsilon > 0$ there is a constant $r_\varepsilon > 0$ such that

$$|g(x, u)| \leq \varepsilon|u| \quad \text{for a.e. } x \in (0, 2\pi) \text{ and all } u \in \mathbb{R} \text{ with } |u| \geq r_\varepsilon. \tag{12}$$

(C3) g satisfies 'sign-like' conditions, i.e., there are functions $A, B \in L^1(0, 2\pi)$ and constants $r < 0 < R$ such that

$$\begin{aligned} g(x, u) \geq A(x) &\quad \text{for a.e. } x \in (0, 2\pi) \text{ and all } u \in \mathbb{R} \text{ with } u \geq R, \\ g(x, u) \leq B(x) &\quad \text{for a.e. } x \in (0, 2\pi) \text{ and all } u \in \mathbb{R} \text{ with } u \leq r. \end{aligned}$$

(C4) Moreover, we assume that the non-homogeneous term $h \in L^1(0, 2\pi)$ satisfies the 'orthogonality-like' conditions

$$\int_0^{2\pi} B(x)\phi_1^*\, dx \leq \int_0^{2\pi} h(x)\phi_1^*\, dx \leq \int_0^{2\pi} A(x)\phi_1^*\, dx, \tag{13}$$

where as aforementioned ϕ_1^* is the eigenfunction associated with the (principal) eigenvalue λ_1 through the dual linear operator.

Before taking up the issue of multiplicity of solutions and the behavior of the solution-set, we first state an existence result for all $\lambda \leq \lambda_0$ (where λ_0 is given by Proposition 2.6), and establish uniform *a priori* bounds when the parameter λ lies in appropriate intervals around zero.

THEOREM 3.1. *Assume that the assumptions (C1)–(C4) hold, then Eq.(9) (or equivalently Eq.(10)) has at least one solution for every* $\lambda \in \mathbb{R}$ *with* $\lambda \leq \lambda_0$. *Moreover, for* $0 < \lambda \leq \lambda_0$, *all solutions are uniformly bounded in* $W^{2,1}(0, 2\pi)$, *independently of* λ.

Recall that *no* multiplicity results occur for Eq.(9) when $g \equiv 0$ and either $\lambda < 0$ or $0 < \lambda \leq \lambda_0$, since the Fredholm alternative argument guarantees uniqueness in this case. We claim that, by somewhat strengthening either (C3) or (C4), we obtain multiplicity results and more importantly describe the behavior of the solution-set. The first result is motivated by the fact that one may allow the equality $A(x) = B(x)$ for a.e. $x \in [0, 2\pi]$ in the assumption (C3). We would like to point that, in this instance, multiplicity may occur *only for one* *value* of λ; more precisely at $\lambda = 0$ (even if $g \not\equiv 0$), with the bifurcation branches in the $(\lambda, |u|_\infty)$-plane being only (semi-infinite) *straight* line rays located on the vertical $|u|_\infty$-axis. It suffices to consider any (nonlinearity) g such that $g(x,u) = 0$ outside a rectangular region $[0,2\pi] \times [-R,R]$. Indeed, for $\lambda = 0$, it is easily seen that the function defined by $u_t := t\phi_1$ is a solution to Eq.(9) for every $t \in \mathbb{R}$ that is such that $|t| \min\limits_{[0,2\pi]}\{\phi_1(x)\} \geq R$; provided $h \equiv 0$ of course. Actually an analysis of the proof of the above existence result (or more precisely, the multiplicity results obtained below) indicates that, provided h is such that $\int_0^{2\pi} h\phi_1^*\, dx = 0$, $\lambda = 0$ is the only parameter-value for which large solutions exist, and the bifurcation from infinity branches are (semi-infinite) straight line rays on the $|u|_\infty$-axis in the $(\lambda, |u|_{C^0([0,2\pi])})$-plane, as described above. Therefore, the bifurcation from infinity parameter-interval collapses to just one-point interval $\{\lambda\} = \{0\}$.

For the rest of the paper, we will be interested in nonlinearities g that satisfy a sign-like condition and that are not identically null outside a compact u-interval in $\mathbb{R}$. In the following result we strengthen somewhat the condition (C3) by requiring strict inequalities (on subsets of $\partial\Omega$ of positive measure) while still retaining the condition (C4).

A simple example to keep in mind here is the (continuous) function g given by $g(x,u) := \eta_+(x)(1+u^2)^{-1}$ for $u \geq R > 0$ and $g(x,u) := -\eta_-(x)(1+u^2)^{-1}$ for $u \leq -r < 0$, where $\eta_\pm \in C^0_P[0,2\pi]$ are nonnegative functions which are positive on subsets of $[0,2\pi]$ of positive measure, or a non-bounded counterpart $g(x,u) := \sqrt[3]{u}\sin^2(u) \pm \eta_\pm(x)(1+u^2)^{-1}$. Here, $A = B = 0$ and $\int_0^{2\pi} h\phi_1^*\, dx = 0$ by (C4). Notice that for the bounded case $\lim\limits_{|u|\to\infty} g(x,u) = 0$ and $\lim\limits_{|u|\to\infty} ug(x,u) = 0$ on $\partial\Omega$, whereas for the unbounded counterpart $\liminf\limits_{u\to\infty} g(x,u) = 0 = \limsup\limits_{u\to-\infty} g(x,u)$ and $\liminf\limits_{u\to\infty} ug(x,u) = 0 = \limsup\limits_{u\to-\infty} ug(x,u)$; that is, no (linear) decay 'rate' at infinity is required. Thus, the terminology (asymptotic) *'very' strong resonance*. Observe also that the so-called Landesman-Lazer condition (see below) fails since one has equality in (C4); however, we are able to 'locate' and 'describe' the solution-branches. The following result is an extension of the main result in [13] to more general linear operators and more general nonlinearities (also see Remark 3.3 below).

Theorem 3.2. *Assume that conditions (C1)–(C2) are met, and that (C3) holds with strict inequalities on subsets of* $[0,2\pi]$ *of positive measure; that is, there are functions* $A, B \in L^1(0,2\pi)$ *and constants* $r < 0 < R$ *such that*

$$\begin{aligned} g(x,u) &> A(x) \quad \text{for a.e. } x \in (0,2\pi) \text{ and all } u \in \mathbb{R} \text{ with } u \geq R, \\ g(x,u) &< B(x) \quad \text{for a.e. } x \in (0,2\pi) \text{ and all } u \in \mathbb{R} \text{ with } u \leq r, \end{aligned}$$

Then, provided (C4) holds, there is a constant $\lambda_- < 0$ *such that, for every* $\varepsilon \in (0, |\lambda_-|)$, *Eq.(9) has at least two solutions, denoted* $(\lambda_\varepsilon^+, u_\varepsilon)$ *and* $(\lambda_\varepsilon^-, v_\varepsilon)$, *with* $-\varepsilon < \lambda_\varepsilon^\pm < 0$ *and*

$$\lim_{\varepsilon\to 0^+} \min\left\{ |u_\varepsilon|_{C^0([0,2\pi])}, |v_\varepsilon|_{C^0([0,2\pi])} \right\} = \infty;$$

that is, they bifurcate from infinity since $\lambda_\varepsilon^\pm \to 0$ *as* $\varepsilon \to 0^+$*.*

Moreover, for $0 \le \lambda \le \lambda_0$*, all solutions (which exist by Theorem 3.1) are uniformly bounded, independently of* λ*. Therefore, bifurcation from infinity occurs only (strictly) to the left of the eigenvalue* λ_1*. (In some sense, the 'strong resonance' conditions 'bend' the bifurcation branches.)*

Remark 3.3. An analysis of the proof of this result will show that the conditions on the nonlinearity g may be replaced by the (slightly) more general (integral) conditions

$$\begin{aligned}
&\int_0^{2\pi} g(x,u)\phi_1^*\,dx > \int_0^{2\pi} A(x)\phi_1^*\,dx \quad \text{for all } u \in \mathbb{R} \text{ with } u \ge R,\\
&\int_0^{2\pi} g(x,u)\phi_1^*\,dx > \int_0^{2\pi} B(x)\phi_1^*\,dx \quad \text{for all } u \in \mathbb{R} \text{ with } u \le R;
\end{aligned}$$

which are in particular fulfilled if the coefficient $b \in AC_P([0,2\pi]) = W_P^{1,1}(0,2\pi)$, and

$$\begin{aligned}
g(x,u) \ge A(x) \quad &\text{for a.e. } x \in (0,2\pi) \text{ and all } u \in \mathbb{R} \text{ with } u \ge R,\\
&\text{with strict inequality on a subset of } (0,2\pi) \text{ of positive measure,}\\
g(x,u) \le B(x) \quad &\text{for a.e. } x \in (0,2\pi) \text{ and all } u \in \mathbb{R} \text{ with } u \le r,\\
&\text{with strict inequality on a subset of } (0,2\pi) \text{ of positive measure,}
\end{aligned}$$

since, in this instance, the conditions on the coefficient b imply that the eigenfunction ϕ_1^* is (strictly) positive on the interval $[0,2\pi]$ by Proposition 2.5.

In the following result we strengthen a little bit the condition (C4) by requiring strict inequalities while keeping (C3) as given. This is the so-called Landsman-Lazer type condition; which has been widely considered in the literature (see e.g. [6]). Again, a simple example to keep in mind here is the (continuous) function g (independent of x) given by $g(u) := \sqrt[3]{u}\,\sin^2(u) + \eta_\pm \tanh(u)$ for $|u| \ge R > 0$ where $\eta_\pm$ are positive numbers with $\eta_- < \eta_+$. Notice that $\liminf_{u\to\infty} g(u) = \eta_+$ and $\limsup_{u\to-\infty} g(u) = -\eta_-$. The following result is an extension of the main result in [12] to more general linear operators and more general nonlinearities (at least as far as periodic solutions are concerned).

Theorem 3.4. *Assume that (C1)–(C3) hold and that*

$$\int_0^{2\pi} g_-(x)\phi_1^*\,dx < \int_0^{2\pi} h(x)\phi_1^*\,dx < \int_0^{2\pi} g_+(x)\phi_1^*\,dx, \tag{14}$$

where $g_+(x) := \liminf_{u\to\infty} g(x,u)$ *and* $g_-(x) := \limsup_{u\to-\infty} g(x,u)$.

Then there is a constant $\lambda_- < 0$ *such that, for every* $\varepsilon \in (0, |\lambda_-|)$, *Eq.(9) has at least two solutions, denoted* $(\lambda_\varepsilon^+, u_\varepsilon)$ *and* $(\lambda_\varepsilon^-, v_\varepsilon)$, *with* $-\varepsilon < \lambda_\varepsilon^\pm < 0$ *and*

$$\lim_{\varepsilon\to 0^+} \min\left\{ |u_\varepsilon|_{C^0([0,2\pi])}, |v_\varepsilon|_{C^0([0,2\pi])} \right\} = \infty;$$

that is, they bifurcate from infinity since $\lambda_\varepsilon^\pm \to 0$ *as* $\varepsilon \to 0^+$.

Moreover, for $0 \le \lambda \le \lambda_0$, *all solutions (which exist by Theorem 3.1) are uniformly bounded, independently of* λ. *Again, bifurcation from infinity occurs only (strictly) to the left of the eigenvalue* λ_1.

Now, we take up the case when the nonlinearity g may have (asymptotically) infinitely many (discrete-countable) zeros (i.e. a sign-like condition with 'oscillation'). In this instance, we strengthen a little bit the condition on the coefficient function b. Therefore, for the sake of clarity, we first state the result for the case when $A = B = 0$; which again implies that the condition (C4) is equivalent to saying that $\int_0^{2\pi} h\phi_1^*\,dx = 0$. The function to keep in mind here is for instance $g(x,u) = \eta_\pm u^{-1}\sin^2(u)$ for $|u| \ge R > 0$ where $\eta_\pm$ are positive numbers, or an unbounded counterpart $g(x,u) = \eta_\pm \sqrt[3]{u}\,\sin^2(u)$ for $|u| \ge R > 0$. Therefore, we consider functions which satisfy a sign condition, vanish asymptotically at discrete-countably many points going to infinity, and have a strict sign in-between them.

Theorem 3.5. *Let the coefficient* b *be such that* $b \in AC_P([0,2\pi]) = W_P^{1,1}(0,2\pi)$. *Assume that conditions (C1) and (C2) are met. Suppose there are sequences of real numbers* $0 >> r_k > r_{k+1} \to -\infty$ *and* $0 << R_k < R_{k+1} \to \infty$ *as* $k \to \infty$ *such that for all* $k \in \mathbb{N}$,

$$g(x, r_k) = 0 \quad \text{and} \quad g(x, R_k) = 0 \quad \text{for a.e. } x \in (0,2\pi) \text{ and}$$

$$\begin{aligned} g(x,u) > 0 &\quad \text{for a.e. } x \in (0,2\pi) \text{ and all } u \in \mathbb{R} \text{ with } R_k < u < R_{k+1}, \\ g(x,u) < 0 &\quad \text{for a.e. } x \in (0,2\pi) \text{ and all } u \in \mathbb{R} \text{ with } r_{k+1} < u < r_k. \end{aligned}$$

Then, provided h *is* $L^1(0,2\pi)$ *with* $\displaystyle\int_0^{2\pi} h\phi_1^*\,dx = 0$, *there is a constant* $\lambda_- < 0$ *such that, for every* $\varepsilon \in (0,|\lambda_-|)$, *Eq.(9) has at least two solutions, denoted* $(\lambda_\varepsilon^+, u_\varepsilon)$ *and* $(\lambda_\varepsilon^-, v_\varepsilon)$, *with* $-\varepsilon < \lambda_\varepsilon^\pm < 0$ *and*

$$\lim_{\varepsilon\to 0^+} \min\left\{ |u_\varepsilon|_{C^0([0,2\pi])}, |v_\varepsilon|_{C^0([0,2\pi])} \right\} = \infty;$$

that is, they bifurcate from infinity since $\lambda_\varepsilon^\pm \to 0$ *as* $\varepsilon \to 0^+$.

Moreover, for $0 < \lambda \le \lambda_0$, *all solutions (which exist by Theorem 3.1) are uniformly bounded, independently of* λ. *Therefore, bifurcation continua from infinity occur to the left of the eigenvalue* λ_1.

A visual rendition sketch for the case when g is as in the above example is given below. (For example, $g(x,u) = \eta_{\pm} u^{-1} \sin^2(u)$ when $|u| \geq R$ with $c = 0$ and $h = 0$.)

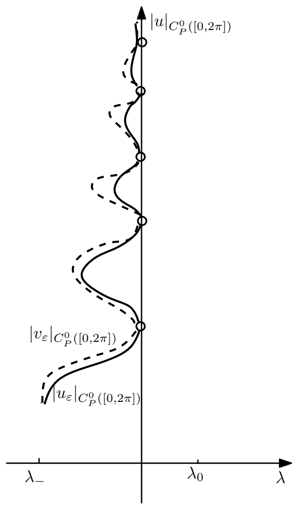

Figure 1: Bifurcation diagram in the case of a 'bounce-off' oscillatory nonlinearity.

Remark 3.6. *(Existence of a third solution)* Let us mention that by using a consequence of the Leray-Schauder Homotopy Continuation Theorem or the so-called Wyburn Lemma (see e.g. [8, 12, 13, 10]), one can show that there is $\lambda_-^* < 0$ with $\lambda_- < \lambda_-^*$ such that for every $\varepsilon \in (0, |\lambda_-^*|)$, one has a third solution w_ε in Theorems 3.2 and 3.4. (The (uniform) bound of these third solutions could for instance be twice the uniform *a-priori* bound obtained for all solutions in the homotopy.)

Remark 3.7. Let us finally point out that one can reverse the inequalities in the conditions (C3)-(C4) appropriately to get results similar to all the ones above. In which case, multiplicity and bifurcation from infinity occur (for λ in a nontrivial interval) to the right of λ_1 *only*, whereas solutions are uniformly bounded on bounded λ-intervals to the left of λ_1. The reader can easily carry out the details.

4. Abstract Setting and *a priori* Bounds

In this section we formulate the problem (9) in an abstract setting. We then proceed to establish *a priori* bounds in $W^{2,1}(0,2\pi)$ for all possible solutions. For that purpose we define the linear operator

$$L : W_P^{2,1}(0,2\pi) \Subset C^0([0,2\pi]) \subset L^1(0,2\pi) \to L^1(0,2\pi) \quad \text{by}$$

$$Lu := u'' + b(x)u' + c(x)u + \lambda_1 u,$$

where $W_P^{2,1}(0,2\pi) \Subset C^0([0,2\pi])$ denotes the compact imbedding of $W_P^{2,1}(0,2\pi)$ in $C^0([0,2\pi])$ (see e.g. [1]). Next, we define the nonlinear (Nemytskĭi) superposition operator

$$\mathcal{N} : C^0([0,2\pi]) \to L^1(0,2\pi) \quad \text{by} \quad \mathcal{N}u = g(\cdot, u(\cdot)).$$

Eq.(9) is then equivalent to

$$Lu + \lambda u + \mathcal{N}u = h, \quad u \in \mathrm{Dom}(L) := W_P^{2,1}(0,2\pi). \tag{15}$$

Now, we shall establish an *a priori* bound for all possible solutions of Eq.(9) or equivalently Eq.(15).

Proposition 4.1. *Assume that the assumptions (C1)–(C4) hold true. Let $\lambda_0 \in \mathbb{R}$ with $\lambda_0 > 0$ be a fixed constant given in Proposition 2.6. Then, there is a constant $R_0 := R_0(\lambda_0) > 0$ such that all possible solutions of Eq.(9) (or equivalently Eq.(15)) with $0 < \lambda \le \lambda_0$ satisfy*

$$|u|_{W^{2,1}(0,2\pi)} \le R_0.$$

That is, all possible solutions of Eq.(15) are (uniformly) bounded in $W^{2,1}(0,2\pi)$ independently of λ, provided $0 < \lambda \le \lambda_0$.

Proof. Suppose that all (possible) solutions in $W_P^{2,1}(0,2\pi)$ are not uniformly bounded in $W^{2,1}(0,2\pi)$. Then, there are sequences $\{\lambda_n\} \subset (0,\lambda_0]$ and $\{u_n\} \subset W_P^{2,1}(0,2\pi)$ with $|u_n|_{W^{2,1}(0,2\pi)} \ge n$ for all $n \in \mathbb{N}$ such that

$$u_n'' + b(x)u_n' + c(x)u_n + \lambda_1 u_n + \lambda_n u_n + g(x,u_n) = h(x) \quad \text{a.e. in } (0,2\pi). \tag{16}$$

Letting $v_n := u_n/|u_n|_{W^{2,1}(0,2\pi)}$, one has that $|v_n|_{W^{2,1}(0,2\pi)} = 1$, and $v_n \in W_P^{2,1}(0,2\pi)$ satisfies

$$\begin{aligned} v_n'' + b(x)v_n' + c(x)v_n + \lambda_1 v_n + \lambda_n v_n + \frac{g(x,u_n)}{|u_n|_{W^{2,1}(0,2\pi)}} \\ = \frac{h(x)}{|u_n|_{W^{2,1}(0,2\pi)}} \text{ a.e. in } (0,2\pi). \end{aligned} \tag{17}$$

Notice that, by the fact that the function g is $L^1(0,2\pi)$-Carathéodory and the sublinear growth condition (12) with $\varepsilon = 1$ e.g., one has that the sequence $\left\{g(\cdot,u_n(\cdot))/|u_n|_{W^{2,1}(0,2\pi)}\right\}$ is bounded in $L^1(0,2\pi)$ since there is a function $\gamma_1 \in L^1(0,2\pi)$ such that $|g(x,u)| \le |u| + \gamma_1(x)$ for a.e. $x \in (0,2\pi)$ and all $u \in \mathbb{R}$. Therefore, since $W_P^{2,1}(0,2\pi)$ is continuously imbedded into $C_P^1([0,2\pi])$, there is a constant $C_1 > 0$ (independent of n) such that

$$|g(x,u_n(x))|/|u_n|_{W^{2,1}(0,2\pi)} \le |v_n(x)| + |\gamma_1(x)|/|u_n|_{W^{2,1}(0,2\pi)} \le C_1 + |\gamma_1(x)|, \quad (18)$$

$|b(x)v_n'(x)| \le C_1|b(x)|$, and $|c(x)v_n(x)| \le C_1|c(x)|$ for a.e. $x \in (0,2\pi)$ and all $n \in \mathbb{N}$. Moreover, since $\lambda_n \in (0,\lambda_0]$ and $W_P^{2,1}(0,2\pi)$ is compactly imbedded into $W_P^{1,1}(0,2\pi)$, one has (by going to subsequences relabeled ($\{\lambda_n\}$ and $\{v_n\}$, if need be) that there exist a number $\mu_0 \in [0,\lambda_0]$ and a function $v \in W_P^{1,1}(0,2\pi)$ such that $\lambda_n \to \mu_0$ and $v_n \to v$ in $W_P^{1,1}(0,2\pi)$ as $n \to \infty$; which implies (for a subsequence similarly relabeled if need be) that $v_n(x) \to v(x)$ and $v_n'(x) \to v'(x)$ for a.e. $x \in (0,2\pi)$ (see e.g. [1, Theorem 4.9]). By using the first inequality in (18), we deduce that $g(x,u_n(x))/|u_n|_{W^{2,1}(0,2\pi)} \to 0$ as $n \to \infty$ for a.e. $x \in (0,2\pi)$ where $v(x) = 0$. Observe that $u_n(x) \to \infty$ if $v(x) > 0$ and $u_n(x) \to -\infty$ if $v(x) < 0$. Therefore, for a.e. $x \in (0,2\pi)$ such that $v(x) \neq 0$, (considering n sufficiently large if need be) we write the quotient $g(x,u_n(x))/|u_n|_{W^{2,1}(0,2\pi)}$ in the form

$$\frac{g(x,u_n(x))}{|u_n|_{W^{2,1}(0,2\pi)}} = \left(\frac{g(x,u_n(x))}{u_n(x)}\right) v_n(x) \to 0 \cdot v(x) = 0 \text{ as } n \to \infty,$$

by the sublinear condition (C2). Thus, in either case one has that the sequence $g(x,u_n(x))/|u_n|_{W^{2,1}(0,2\pi)} \to 0$ as $n \to \infty$ for a.e. $x \in (0,2\pi)$. By the Lebesgue Dominated Convergence Theorem, it follows that $b(\cdot)v_n' \to b(\cdot)v$, $c(\cdot)v_n \to c(\cdot)v$ and $g(\cdot,u_n(\cdot))/|u_n|_{W^{2,1}(0,2\pi)} \to 0$ in $L^1(0,2\pi)$ as $n \to \infty$.

Now, by using Eq.(17), we deduce that $v_n'' \to -b(x)v - c(x)v - \lambda_1 v - \mu_0 v$ in $L^1(0,2\pi)$ with $v_n \to v$ in $W_P^{1,1}(0,2\pi)$ as $n \to \infty$ and $\mu_0 \in [0,\lambda_0]$. The (strong) closedness of the differentiation-operator from $W_P^{1,1}(0,2\pi)$ into $L^1(0,2\pi)$ implies that $v \in W_P^{2,1}(0,2\pi)$ and that $v_n \to v$ in $W_P^{2,1}(0,2\pi)$ as $n \to \infty$ with $v'' = -b(x)v' - c(x)v - \lambda_1 v - \mu_0 v$ for a.e. $x \in (0,2\pi)$; that is,

$$Lv + \mu_0 v = 0. \quad (19)$$

It follows from Proposition 2.6 that either $v(x) > 0$ on $[0,2\pi]$ or $v(x) < 0$ on $[0,2\pi]$ since $|v|_{W^{2,1}(0,2\pi)} = 1$. Using the duality pairing (see e.g. [1]), we get that

$$0 = \langle Lv + \mu_0 v, \phi_1^* \rangle = \langle v, L^*(\phi_1^*) \rangle + \mu_0 \int_0^{2\pi} v\phi_1^* \, dx = \mu_0 \int_0^{2\pi} v\phi_1^* \, dx,$$

since ϕ_1^* is an eigenfunction of the adjoint L^* associated with the eigenvalue zero. This implies that $\mu_0 = 0$ since ϕ_1^* is a nonnegative eigenfunction and $|v(x)| > 0$ on $[0, 2\pi]$. Therefore, $Lv = 0$; i.e., $v = t\phi_1$ for some real constant $t \neq 0$ since λ_1 is simple.

In what follows, we assume without loss of generality that $v(x) > 0$ on $[0, 2\pi]$; i.e., $t > 0$ (the case $v(x) < 0$ can be treated in a similar way). This implies that there is a constant $\epsilon_0 > 0$ such that $v(x) = t\phi_1(x) \geq \epsilon_0$ for all $x \in [0, 2\pi]$ since the eigenfunction ϕ_1 of L is (strictly) positive on $[0, 2\pi]$.

Since $v_n \to v$ uniformly on $[0, 2\pi]$, one has that $u_n(\cdot) = v_n(\cdot)|u_n|_{W^{2,1}(0,2\pi)} \to \infty$ uniformly on $[0, 2\pi]$. Therefore, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ one has that

$$u_n(x) \geq R \quad \text{for all } x \in [0, 2\pi], \tag{20}$$

where $R > 0$ is the constant given in the assumption (C3). Now, using again the duality pairing in Eq.(16), we deduce that $\langle Lu_n + \lambda_n u_n + \mathcal{N}u_n, \phi_1^* \rangle = \int_0^{2\pi} h\phi_1^*\,dx$; i.e., $\langle u_n, L^*(\phi_1^*)\rangle + \lambda_n \int_0^{2\pi} u_n\phi_1^*\,dx + \int_0^{2\pi} g(x, u_n)\phi_1^*\,dx = \int_0^{2\pi} h\phi_1^*\,dx$.

Since $0 < \lambda_n \leq \lambda_0$, it follows from Eq.(16), the inequality (20) and the assumption (C3) that for each $n \geq n_0$,

$$\begin{aligned} 0 > -\lambda_n \int_0^{2\pi} u_n\phi_1^*\,dx &= \int_0^{2\pi} g(x, u_n)\phi_1^*\,dx - \int_0^{2\pi} h(x)\phi_1^*\,dx \\ &\geq \int_0^{2\pi} A(x)\phi_1^*\,dx - \int_0^{2\pi} h(x)\phi_1^*\,dx; \end{aligned}$$

that is,

$$\int_0^{2\pi} h(x)\phi_1^*\,dx > \int_0^{2\pi} A(x)\phi_1^*\,dx;$$

which is a contradiction with the second inequality in the assumption (C4). Therefore, all possible solutions of Eq.(9) (or equivalently Eq.(15)) are (uniformly) bounded in $W^{2,1}(0, 2\pi) \subset C^0([0, 2\pi])$ independently of λ, provided that $0 < \lambda \leq \lambda_0$. The proof is complete. ☐

Let us mention that a similar result holds for all λ negative (and bounded away from zero). More precisely, we have the following uniform *a priori* bound.

PROPOSITION 4.2. *Let $\alpha_0, \alpha_1 \in \mathbb{R}$ be (fixed negative) constants such that $-\infty < \alpha_0 < \alpha_1 < 0$. Suppose that the assumptions (C1)–(C2) hold. Then, there exists a constant $R_0 := R_0(\alpha_0, \alpha_1) > 0$ such that all possible solutions of Eq.(9), with $\alpha_0 \leq \lambda \leq \alpha_1$, satisfy*

$$|u|_{W^{2,1}(0,2\pi)} \leq R_0.$$

That is, all possible solutions of Eq.(9) (or equivalently Eq.(15)) are (uniformly) bounded in $W^{2,1}(0, 2\pi)$ independently of λ, provided that $\alpha_0 \leq \lambda \leq \alpha_1 < 0$.

The proof is similar to the one above up to Eq.(19) where now $\mu_0 \in [\alpha_0, \alpha_1]$. However, since $\alpha_1 < 0$, it follows that $\mu_0 \leq \alpha_1 < 0$ is in the resolvent of L (see e.g. the second part of Remark 2.7), and hence $v \equiv 0$ on $[0, 2\pi]$. This is a contradiction with the fact that $|v|_{W^{2,1}(0,2\pi)} = 1$. Therefore, all possible solutions of Eq.(15) (or equivalently Eq.(9)) are (uniformly) bounded in $W^{2,1}(0, 2\pi)$ independently of λ, provided that $\alpha_0 \leq \lambda \leq \alpha_1$. The proof is complete.

5. Proofs of main results

In this section we prove the main results by using the topological degree theory, continuation methods and bifurcation from infinity techniques. We first prove the existence part of the results, and then proceed to show multiplicity and bifurcation.

Proof of Theorem 3.1. First we consider the case when $\lambda \geq 0$ is fixed. Picking $\delta \in \mathbb{R}$ such that $0 < \delta < \lambda_0$, and following the notation of the previous section, we consider the homotopy

$$Lu + \delta u + \theta[(\lambda - \delta)u + \mathcal{N}u] = \theta h, \quad u \in \mathrm{Dom}(L), \tag{21}$$

where $\theta \in [0, 1)$; which, when $\theta = 0$, reduces to the homogeneous linear problem $Lu + \delta u = 0$ that has only the trivial solution; for otherwise, Proposition 2.6 and an argument similar to that used after Eq.(19) would imply that $\delta = 0$. Since the linear operator $L + \delta I$ defined by $L + \delta I : W^{2,1}_P(0, 2\pi) \to L^1(0, 2\pi)$ is bounded, one-to-one and onto (see e.g. the arguments used in the proof of Lemma 2.3), it follows that (21) is equivalent to the fixed point homotopy

$$u = \theta(L + \delta I)^{-1} \left((\delta - \lambda)Iu - \mathcal{N}u + h\right), \quad u \in \mathrm{Dom}(L). \tag{22}$$

Therefore, by the compactness of the imbedding $W^{2,1}_P(0, 2\pi)$ into $L^1(0, 2\pi)$ and the topological degree theory (see e.g. [10]), it suffices to show that all possible solutions of the homotopy (22) are bounded in $W^{2,1}(0, 2\pi)$, independently of $\theta \in [0, 1)$, in order to conclude that Eq.(22) has at least one solution for $\theta = 1$ as well.

Indeed, observing that $0 < (1 - \theta)\delta + \theta\lambda \leq \max\{\lambda, \delta\} \leq \lambda_0$ for $0 \leq \theta < 1$, it follows from Proposition 4.1 that all possible solutions of Eq.(21) (or equivalently Eq.(22)) are (uniformly) bounded in $W^{2,1}(0, 2\pi)$ independently of $\theta \in [0, 1)$. This proves the first part of Theorem 3.1. The second part of Theorem 3.1 follows readily from Proposition 4.1.

To prove the existence of at least one solution for $\lambda < 0$ (fixed), we consider the homotopy (21) where $\delta < 0$ and now $\theta \in [0, 1]$. (Notice that $\theta = 1$ is included here.) Observing that $\alpha_0 := \min\{\lambda, \delta\} \leq (1-\theta)\delta + \theta\lambda \leq \max\{\lambda, \delta\} := \alpha_1 < 0$ for $0 \leq \theta \leq 1$, it follows from Proposition 4.2 that all possible solutions of Eq.(21) (or equivalently Eq.(22)) are (uniformly) bounded in $W^{2,1}(0,2\pi)$ independently of $\theta \in [0,1]$. The existence of at least one solution for each $\theta \in [0,1]$ follows from topological degree arguments as above. (It should be noted that Assumptions (C3)–(C4) do not matter when $\lambda < 0$, at least as far as the existence of at least one solution is concerned.) The proof is complete. $\square$

Now, we take up the issue of multiplicity and bifurcation (from infinity) of solutions for λ "near" zero; actually λ to the left of zero as it will be seen.

Proof of Theorem 3.2. We first show that all possible solutions of Eq.(15) are (uniformly) bounded in $W_P^{2,1}(0,2\pi)$ when $\lambda = 0$ as well; that is, the conclusion of Theorem 3.1 actually holds true for all $\lambda \in [0, \lambda_0]$. Indeed the proof is similar to that of Theorem 3.1 except that we consider the homotopy-parameter $\theta \in [0,1]$. Therefore, it suffices to show that all possible solutions of the homotopy (22) are bounded in $W^{2,1}(0,2\pi)$ for $\theta = 1$ and $\lambda = 0$ as well. For that purpose, we follow the arguments in the proof of Proposition 4.1 with $\lambda = 0$ up to the inequality (20). Now, using the duality pairing with the eigenfunction ϕ_1^* in Eq.(16) (recall that $\theta = 1$ and $\lambda = 0$), and the fact that ϕ_1^* is an eigenfunction of L^*, it follows from Eq.(16), the inequality (20) and the (stronger) assumption on the functions g and A in Theorem 3.2 that for each $n \geq n_0$,

$$0 = \int_0^{2\pi} g(x, u_n)\phi_1^* \, dx - \int_0^{2\pi} h\phi_1^* \, dx > \int_0^{2\pi} A(x)\phi_1^* \, dx - \int_0^{2\pi} h(x)\phi_1^* \, dx;$$

that is,

$$\int_0^{2\pi} h(x)\phi_1^* \, dx > \int_0^{2\pi} A(x)\phi_1^* \, dx.$$

This is a contradiction with the second inequality in the assumption (C4). Hence, all possible solutions of Eq.(15) are (uniformly) bounded in $W_P^{2,1}(0,2\pi)$ for $\lambda = 0$ as well. Thus, in this case, one gets the boundedness of all possible solutions in $W_P^{2,1}(0,2\pi)$ as in Proposition 4.1.

Now, we proceed to look into the situation regarding multiplicity and bifurcation from infinity. As in the proof of Theorem 3.1, we let $\delta \in \mathbb{R}$ be sufficiently small such that $0 < \delta < \lambda_0$, and observe that Eq.(15) is equivalent to the fixed point equation

$$u = (\delta - \lambda)(L + \delta I)^{-1}u - (L + \delta I)^{-1}\left(\mathcal{N}u - h\right).$$

Setting

$$\mu := \delta - \lambda, \; Hu := (L + \delta I)^{-1}u \text{ and } Ku := -(L + \delta I)^{-1}\left(\mathcal{N}u - h\right),$$

it follows that the above fixed point equation is equivalent to the equation

$$u = \mu H u + K(u), \; u \in C_P^0([0,2\pi]). \tag{23}$$

Notice that Eq.(23) has now an abstract form considered e.g. in [17] for bifurcation from infinity purposes. From this setup, it follows that, when $\lambda = 0$, the constant $\mu^{-1} = \delta^{-1}$ is the principal eigenvalue of H and that, by the compact imbedding of $W_P^{2,1}(0, 2\pi)$ into $C_P^0([0, 2\pi])$, the solution-map

$$H := (L + \delta I)^{-1} : C_P^0([0, 2\pi]) \to W_P^{2,1}(0, 2\pi) \overset{c}{\hookrightarrow} C_P^0([0, 2\pi])$$

is a compact linear operator when considered as an operator from $C_P^0([0, 2\pi])$ into $C_P^0([0, 2\pi])$. Since by the Carathédory condition (C1) the superposition operator $\mathcal{N} : C_P^0([0, 2\pi]) \to L^1(0, 2\pi)$ (defined by $\mathcal{N}(u) := g(\cdot, u(\cdot))$) is continuous (by e.g. using Lebesgue Dominated Convergence Theorem) and $h \in L^1(0, 2\pi)$, one has that $\mathcal{N}(\cdot) + h$ maps $C_P^0([0, 2\pi])$ continuously into $L^1(0, 2\pi)$. Therefore

$$K : C_P^0([0, 2\pi]) \to W_P^{2,1}(0, 2\pi) \overset{c}{\hookrightarrow} C_P^0([0, 2\pi])$$

is a completely continuous mapping when viewed as a nonlinear operator from $C_P^0([0, 2\pi])$ into $C_P^0([0, 2\pi])$.

Now, we wish to show that $K(u) = \mathrm{o}(|u|_{C_P^0([0,2\pi])})$ as $|u|_{C_P^0([0,2\pi])} \to \infty$. Let us set $w = K(u)$ for $u \in C_P^0([0, 2\pi])$; that is, $w \in W_P^{2,1}(0, 2\pi)$ satisfies the operator equation $(L + \delta I)w = -\mathcal{N}(u) + h$ for $u \in C_P^0([0, 2\pi])$. By the arguments similar to those used in the proof of Lemma 2.2, there is a constant $C_1 > 0$ (independent of u) such that

$$|w|_{W_P^{2,1}(0,2\pi)} \le C_1 \left(|g(\cdot, u(\cdot))|_{L^1(0,2\pi)} + |h|_{L^1(0,2\pi)}\right). \tag{24}$$

Using the sublinear growth condition (C2), we first proceed to show that the real-valued function $|g(\cdot, u(\cdot))|_{L^1(0,2\pi)}$ is a $\mathrm{o}(|u|_{C_P^0([0,2\pi])})$ as $|u|_{C_P^0([0,2\pi])} \to \infty$. Indeed, let $\varepsilon > 0$ be given, it follows from the Carathéodory condition (C1) and the sublinearity assumption (C2) that there exist a constant $r_\varepsilon > 0$ and a function $a_\varepsilon \in L^1(0, 2\pi) \setminus \{0\}$ such that for every $u \in C_P^0([0, 2\pi])$ one has

$$|g(x, u(x))| \le \frac{\varepsilon}{2}|u(x)| \le \frac{\varepsilon}{2}|u|_{C_P^0([0,2\pi])} \quad \text{a.e. where} \quad |u(x)| \ge r_\varepsilon,$$

and

$$|g(x, u(x))| \le |a_\varepsilon(x)| \quad \text{a.e. where} \quad |u(x)| \le r_\varepsilon.$$

Picking $R_\varepsilon := R(\varepsilon) \ge 2|a_\varepsilon|_{L^1(0,2\pi)}/\varepsilon$, it follows that for $|u|_{C_P^0([0,2\pi])} \ge R_\varepsilon$ one has

$$|g(\cdot, u(\cdot))|_{L^1(0,2\pi)}/|u|_{C^0([0,2\pi])} \le \varepsilon. \tag{25}$$

This shows that for every $\varepsilon > 0$ there is a constant $R_\varepsilon > 0$ such that the inequality (25) holds provided $|u|_{C_P^0([0,2\pi])} \ge R_\varepsilon$; that is, $|g(\cdot, u(\cdot))|_{L^1(0,2\pi)} = \mathrm{o}(|u|_{C_P^0([0,2\pi])})$ as $|u|_{C_P^0([0,2\pi])} \to \infty$; which by using the inequality (24) implies that $|w|_{W^{2,1}(0,2\pi)} = \mathrm{o}(|u|_{C_P^0([0,2\pi])})$ as $|u|_{C_P^0([0,2\pi])} \to \infty$. Since $W_P^{2,1}(0, 2\pi)$

is continuously imbedded into $C^0_P([0,2\pi])$ and $w = K(u)$, it follows that $|K(u)|_{C^0_P([0,2\pi])} = \mathrm{o}(|u|_{C^0_P([0,2\pi])})$ for $|u|_{C^0_P([0,2\pi])} \to \infty$; as needed.

Therefore, $\lambda = 0$ is a bifurcation point from infinity since all assumptions of the bifurcation from infinity result are fulfilled (see e.g. [17, p. 465, Theorem 1.6 and Corollary 1.8], also see [20, 2]); that is, there exist two connected sets of solutions $\mathcal{C}^+$, $\mathcal{C}^- \subset \mathbb{R} \times C^0_P([0,2\pi])$ with $\mathcal{C}^+ \cap \mathcal{C}^- = \emptyset$ which are such that for every (sufficiently) small $\varepsilon > 0$, $\mathcal{C}^+ \cap U_\varepsilon \neq \emptyset$, $\mathcal{C}^- \cap U_\varepsilon \neq \emptyset$ where $U_\varepsilon := \Big\{(\lambda, u) \in \mathbb{R} \times C^0_P([0,2\pi]) : |\lambda| < \varepsilon, |u|_{C^0_P([0,2\pi])} > 1/\varepsilon\Big\}$. (Observe that, by the regularity of solutions, $u \in W^{2,1}_P(0,2\pi)$ since it is a solution of the fixed point equation (23).)

Now, since all 2π-periodic solutions are uniformly bounded in $W^{2,1}(0,2\pi)$ for all $\lambda \in [0,\lambda_0]$ (see Proposition 4.1 and the above bound in the case $\lambda = 0$) and for all $\lambda \in [\alpha_0, \alpha_1]$ with $\alpha_1 < 0$ (see Proposition 4.2), there then exists a deleted left-neighborhood of 0 in $\mathbb{R}$; i.e., there is $\lambda_- < 0$, such that for every $\varepsilon > 0$ with $\varepsilon < |\lambda_-|$, there are two distinct solutions $(\lambda^+_\varepsilon, u_\varepsilon) \in C^+$ and $(\lambda^-_\varepsilon, v_\varepsilon) \in C^-$ with $-\varepsilon < \lambda^\pm_\varepsilon < 0$, $u_\varepsilon \neq v_\varepsilon$ and $\min\big\{|u_\varepsilon|_{C^0([0,2\pi])}, |v_\varepsilon|_{C^0([0,2\pi])}\big\} > 1/\varepsilon$. Letting $\varepsilon \to 0^+$, it follows that $\lambda^\pm_\varepsilon \to 0$ and $\min\big\{|u_\varepsilon|_{C^0([0,2\pi])}, |v_\varepsilon|_{C^0([0,2\pi])}\big\} \to \infty$. The proof is complete. □

Proof of Theorem 3.4. As in the proof of Theorem 3.2, we first show that all possible solutions of Eq.(15) are (uniformly) bounded in $W^{2,1}(0,2\pi)$ when $\lambda = 0$ as well; that is, the conclusion of Theorem 3.1 actually holds true for all $\lambda \in [0,\lambda_0]$. As before, the proof is similar to that of Theorem 3.1 except that we consider the homotopy-parameter $\theta \in [0,1]$. Therefore, it suffices to show that all possible solutions of the homotopy (22) are bounded in $W^{2,1}(0,2\pi)$ for $\theta = 1$ and $\lambda = 0$ as well. For that purpose, we follow the arguments in the proof of Proposition 4.1 with $\lambda = 0$ up to the inequality (20). Now, using the duality pairing with the eigenfunction ϕ^*_1 in Eq.(16) and the fact that ϕ^*_1 is an eigenfunction of L^*, it follows from Eq.(16) that for each $n \geq n_0$, $0 = \displaystyle\int_0^{2\pi} g(x,u_n)\phi^*_1\,dx - \int_0^{2\pi} h\phi^*_1\,dx$. The inequality (20), the assumption (C3), and Fatou's lemma imply that

$$\begin{aligned} 0 &= \liminf_{n\to\infty} \int_0^{2\pi} g(x,u_n)\phi^*_1\,dx - \int_0^{2\pi} h\phi^*_1\,dx \\ &\geq \int_0^{2\pi} \liminf_{n\to\infty} g(x,u_n)\phi^*_1\,dx - \int_0^{2\pi} h\phi^*_1\,dx = \int_0^{2\pi} g_+(x)\phi^*_1\,dx - \int_0^{2\pi} h\phi^*_1\,dx; \end{aligned}$$

that is, $\displaystyle\int_0^{2\pi} h\phi^*_1\,dx \geq \int_0^{2\pi} g_+(x)\phi^*_1\,dx$. This is a contradiction with the second inequality in the assumption (14) of Theorem 3.4. Therefore, all possible solutions of Eq.(15) are (uniformly) bounded in $W^{2,1}(0,2\pi)$ for $\lambda = 0$ as well. One can now can proceed as in the proof of Theorem 3.2 to establish multiplicity and bifurcation from infinity. The proof is complete. □

Proof of Theorem 3.5. As in the above proofs, we analyse more carefully the behavior of all possible solutions of Eq.(15) (or equivalently Eq.(9)) when $\lambda = 0$. We first show that all possible *non-constant* solutions of Eq.(15) are (uniformly) bounded in $W^{2,1}(0, 2\pi)$ when $\lambda = 0$. For that purpose, we follow the arguments in the proof of Proposition 4.1 with $\lambda = 0$ up to the inequality (20) with $u_n \neq$ cst for all $n \geq n_0$ and $R = R_1$ is the first element of the sequence $\{R_k\}_{k\geq 1}$ given in the statement of the theorem. Now, using the duality pairing with the eigenfunction ϕ_1^* in Eq.(16) and the fact that ϕ_1^* is an eigenfunction of L^*, it follows from Eq.(16) that for each $n \geq n_0$, $0 = \int_0^{2\pi} g(x, u_n)\phi_1^* \, dx$. By using the (strict) positivity of the eigenfunction ϕ_1^* (see Proposition 2.5), the inequality (20) which implies the non-negativity of $g(\cdot, u_n(\cdot))$ by the assumption in the theorem, we get that $g(\cdot, u_n(\cdot)) \equiv 0$ a.e. on $[0, 2\pi]$. This is a contradiction with the positivity assumption on g in the theorem since $u_n \neq$ constant for all $n \geq n_0$ (i.e., $u_n \not\equiv R_k$ for some k). Thus, all possible *non-constant* solutions of Eq.(15) where $\lambda = 0$ are (uniformly) bounded in $W^{2,1}(0, 2\pi)$. However, in this instance, large (in norm) constant solutions might occur in Eq.(15) when $\lambda = 0$. The above argument shows that if they do occur, then they must necessarily be elements of the sequences $\{R_k\}$ or $\{r_k\}$ of real numbers given in the statement of the theorem (for k large enough).

Since the sequences $\{R_k\}$ and $\{r_k\}$ are discrete sets, and the continua $\mathcal{C}^+$ and $\mathcal{C}^+$ are connected, we deduce as in the proof of Theorem 3.2 that there exists a deleted left-neighborhood of 0 in $\mathbb{R}$; i.e., there is $\lambda_- < 0$, such that for every $\varepsilon > 0$ with $\varepsilon < |\lambda_-|$, there are two distinct solutions $(\lambda_\varepsilon^+, u_\varepsilon) \in C^+$ and $(\lambda_\varepsilon^-, v_\varepsilon) \in C^-$ with $-\varepsilon < \lambda_\varepsilon^\pm < 0$, $u_\varepsilon \neq v_\varepsilon$, $\min\left\{|u_\varepsilon|_{C^0([0,2\pi])}, |v_\varepsilon|_{C^0([0,2\pi])}\right\} > 1/\varepsilon$. It follows that $\lambda_\varepsilon^\pm \to 0$ and $\min\left\{|u_\varepsilon|_{C^0([0,2\pi])}, |v_\varepsilon|_{C^0([0,2\pi])}\right\} \to \infty$ as $\varepsilon \to 0^+$. (Notice that these continua could 'connect' to the discrete set of large constant solutions, if any; i.e, oscillate on the left of $\lambda = 0$ and 'bounce-off' theses discrete constant solutions as $\varepsilon \to 0$!) The proof is complete. □

Remark 5.1. As indicated above, with the coefficient $b \in AC_P([0, 2\pi]) = W_P^{1,1}(0, 2\pi)$, we may replace the condition $A = B = 0$ in Theorem 3.5 by a (slightly) more general condition where $B \leq A$ are possibly nonzero constants. In this case, in addition to assuming that the conditions (C1), (C2) and (C4) are met, we suppose that there exist sequences of real numbers $0 >> r_k > r_{k+1} \to -\infty$ and $0 << R_k < R_{k+1} \to \infty$ as $k \to \infty$ such that for all $k \in \mathbb{N}$,

$$g(x, r_k) = B \quad \text{and} \quad g(x, R_k) = A \quad \text{for a.e. } x \in (0, 2\pi) \text{ and}$$

$$g(x, u) > A \quad \text{for a.e. } x \in (0, 2\pi) \text{ and all } u \in \mathbb{R} \text{ with } R_k < u < R_{k+1},$$
$$g(x, u) < B \quad \text{for a.e. } x \in (0, 2\pi) \text{ and all } u \in \mathbb{R} \text{ with } r_{k+1} < u < r_k.$$

That is, the (bounce-off) oscillations of the nonlinearity occur with respect to the constants A and B. Observe that the condition $\int_0^{2\pi} h\phi_1^* \, dx = 0$ is now replaced by the more general condition (C4). The proof is similar to that of Theorem 3.5.

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