I. Evaluating a double Summation

$$\sum_X \sum_Y X_i p(X_i, Y_i)$$

$y \backslash x$	0	1	
0	$P(0,0)$	$P(0,1)$	$P(Y=0)$
1	$P(1,0)$	$P(1,1)$	$P(X=1)$
	$P(X=0)$	$P(X=1)$	

$y \backslash x$	0	1	
0	1/8	1/8	1/4
1	3/8	3/8	3/4
	1/2	1/2	

First sum over Y

$$\sum_{x_i} \sum_{y_i} X_i p(X_i, Y_i) = [0 * p(0,0) + 0 * p(1,0)] + [1 * p(0,0) + 1 * p(1,0)]$$

$$= [0 * 1/8 + 0 * 3/8] + [1 * 1/8 + 1 * 3/8]$$

$$= [0 * 1/2] + [1 * 1/2]$$

This gives the marginal probabilities of X so we're left with

$$\sum_{x_i} X_i p(X_i) = [0 * 1/2] + [1 * 1/2] \text{ to be summed over x}$$

$$\sum_{x_i} \sum_{y_i} X_i p(X_i, Y_i) = .5$$

II. Using the Standardized normal table

1. Find area between 0 and 1.

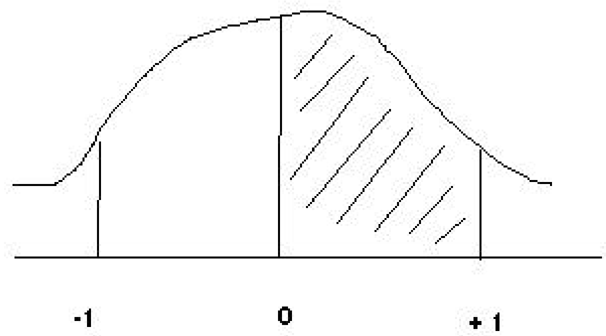


table for z=1.0 $p(0 \leq z \leq 1.0) = .341$
area between 0 and -1 also=.341
2. Find area between 0 and -2

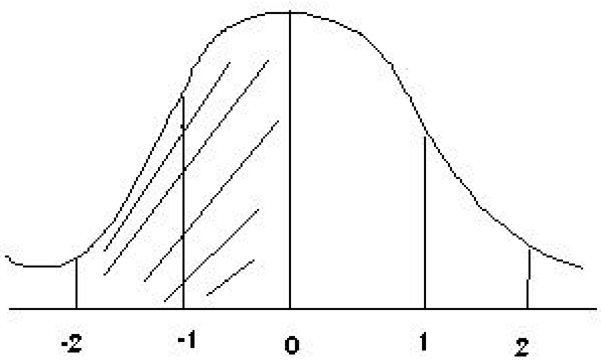
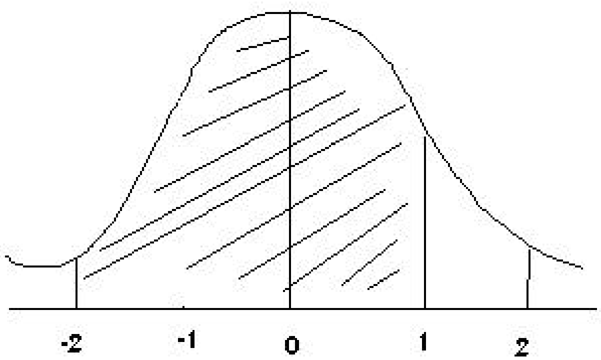


table for z=2.0 $p(0\leq z \leq 2.0) = .477 = 0\leq z \leq -2.0$
note $\neq 2p(0\leq z \leq 1.0)$
3. Find area between -2 and 1
$p(-2\leq z \leq 1.0) = p(0\leq z \leq 1.0) + p(0\leq z \leq 1.0) = .477 + .341 = .818$

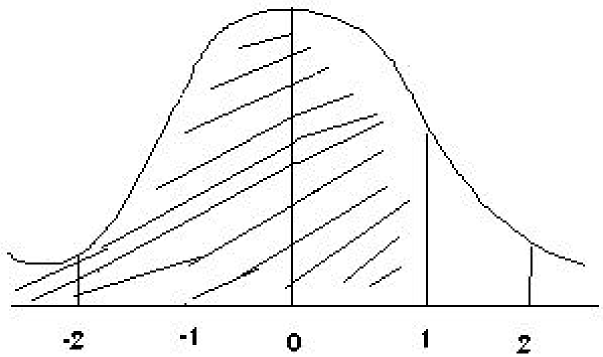


4. Find area to right of +1

$p(1\leq z) = p(0\leq z) - p(0\leq z \leq 1.0)$ but $p(0 \leq z) = .5 = .5 - .341 = .159$

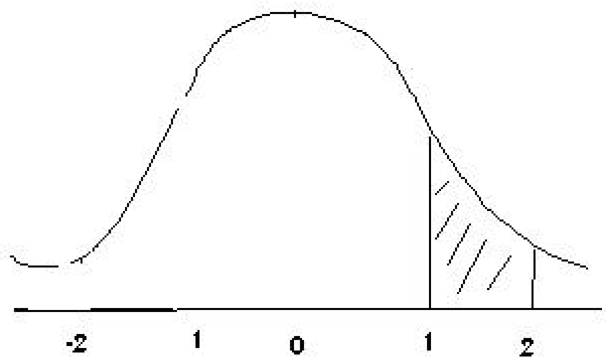
5. Find area to the left of z

$p(z\leq 2) = p(z\leq 0) + p(0\leq z \leq 2.0) = .5 + .477 = .977$



6.Find area between +1 and +2

$p(1 \leq z \leq 2.0) = p(0 \leq z \leq 2.0) - p(0 \leq z \leq 1.0) = .477 - .341 = .136$



III.Estimating the Population Variance from the Sample Variance

We will show below that the sample variance is a biased estimator of the population variance and then that the correction for degrees of freedom, which transform the sample variance to s^2 yields an unbiased estimator of the population variance.

First we develop a useful identity.

$\sum(X_i - \mu)^2$ can be rewritten by adding and subtracting $\bar{X}$ to the term in parenthesis

$$\begin{aligned}
\sum(X_i - \mu)^2 &= \sum(X_i - \bar{X} + \bar{X} - \mu)^2 \\
&= \sum[(X_i - \bar{X})(\bar{X} - \mu)]^2 \\
&= \sum\left[\,(X_i - \bar{X})^2 + 2(X_i - \bar{X})(\bar{X} - \mu) + (\bar{X} - \mu)^2\,\right] \\
&= \sum(X_i - \bar{X})^2 + 2(X_i - \bar{X})(\bar{X} - \mu) + n(\bar{X} - \mu)^2
\end{aligned}$$

Since $\bar{X}$ and μ are constant and can be factored out of the $\sum$ and since $\sum$ of a constant=n * constant. Then note
$\sum(X_i - \bar{X}) = \sum X_i - n\bar{X} = n\bar{X} - n\bar{X} = 0$
so the second term vanishes.

$$\sum(\bar{X} - \mu)^2 = \sum(X_i - \bar{X})^2 + 2(\bar{X} - \mu)\sum(X_i - \bar{X}) + n(\bar{X} - \mu)^2 = \sum(X_i - \bar{X})^2 + n(\bar{X} - \mu)$$

We can also rearrange with equation so:
$\sum(X_i - \bar{X}) = \sum(\bar{X} - \mu)^2 - n(\bar{X} - \mu)$
Now let us examine the expected value of the sample variance.

$$E\left[\frac{\sum(X_i - \bar{X})^2}{n}\right] = \frac{1}{n}E\left[\sum(X_i - \bar{X})^2\right] = \frac{1}{n}E\left[\sum(\bar{X} - \mu)^2 - n(\bar{X} - \mu)^2\right]$$
$$= \frac{1}{n}\sum E[(\bar{X} - \mu)]^2 - nE\left[(\bar{X} - \mu)^2\right]$$

Since the expected values of a sum of random variables is the sum of the expected values of the variables.
Note that the first term expected value is just the expression for the variance of x and the second term expected value is just the expression for the variance of $\bar{X}$ (recalling that $E(\bar{X}) = \mu$. So

$$E\left[\frac{\sum(X_i - \bar{X})^2}{n}\right] = \frac{1}{n}\left(\sum \sigma_x^2 - n\sigma_{\bar{x}}^2\right) = \frac{1}{n}\left(n\sigma_x^2 - \frac{n\sigma_{\bar{x}}^2}{n}\right) = \sigma_x^2\frac{n-1}{n}$$

Thus $\frac{\sum(X_i - \bar{X})^2}{n}$ is a biased estimator by the amount $\frac{n-1}{n}$.If however we multiply $\frac{\sum(X_i - \bar{X})^2}{n}$ by $\frac{n}{n-1}$ and take the expected value, we get

$$E\left[\frac{\sum(X_i - \bar{X})^2}{n} * \frac{n}{n-1}\right] = \frac{n}{n-1}E\left[\frac{\sum(X_i - \bar{X})^2}{n}\right] = (\frac{n}{n-1})\sigma_x^2(\frac{n-1}{n}) = \sigma_x^2$$

So $S_x^2 = \frac{\sum(X_i - \bar{X})^2}{n} * \frac{n}{n-1} = \frac{\sum(X_i - \bar{X})^2}{n-1}$
is an unbiased estimator of the population variance.
$E(S_x^2) = \sigma^2$