# I.Expected Values

Denote the expected value of the function g($x_i$) as E[$g(x_i)$].
E[$g(x_i)$] $= \sum_{i=-\infty}^{\infty} g(x_i)p(x_i)$ where $p(x_i)$ is the probability of the occurrence of the value $x_{i.}$
Consider the data set below, indexing Y values on i, and observation numbers on j. Then j=1,...,10 i=1,2,3,4,5

| *obs#* | $Y_j$ | $Y_j - \bar{Y}$ | $(Y_j - \bar{Y})^2$ |
|---|---|---|---|
| 1 | 2 | −1 | 1 |
| 2 | 3 | 0 | 0 |
| 3 | 1 | −2 | 4 |
| 4 | 3 | 0 | 0 |
| 5 | 4 | 1 | 1 |
| 6 | 3 | 0 | 0 |
| 7 | 5 | −2 | 4 |
| 8 | 4 | 1 | 1 |
| 9 | 2 | −1 | 1 |
| 10 | 3 | 0 | 0 |
| *Total* | 30 | 0 | 12 |

$$\frac{\sum_j Y_j}{N} = \frac{2+3+1+3+4+3+5+4+2+3}{10} = \frac{30}{10} = 3 = \bar{Y}$$

$$regrouping = \frac{1}{10} + \left(\frac{2}{10} + \frac{2}{10}\right) + \left(\frac{3}{10} + \frac{3}{10} + \frac{3}{10} + \frac{3}{10}\right) + \left(\frac{4}{10} + \frac{4}{10}\right) + \left(\frac{5}{10}\right) = \frac{30}{10} = 3 = \bar{Y}$$

$$\frac{\sum_j Y_j - \bar{Y}}{N} = \frac{1+0+4+0+1+0+4+1+1+0}{10} = \frac{12}{10} = 1.2$$

$$regrouping = \left(\left(\frac{0}{10} + \frac{0}{10} + \frac{0}{10} + \frac{0}{10}\right) + \left(\frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10}\right) + \left(\frac{4}{10} + \frac{4}{10}\right)\right) = \frac{12}{10} = 1.2$$

Let assume that the relative frequencies f($Y_i$)$/N$ in the data above reflect the relative population frequencies and therefore the probabilities. The data can be presented as follows:

| $Y_i$ | $f(Y_i)/N$ | $p(Y_i)$ | $Y_i \bullet p(Y_i)$ | $Y_i - E(Y)$ | $[Y_i - E(Y)]^2$ | $[Y_i - E(Y)]^2 \bullet p(Y_i)$ |
|---|---|---|---|---|---|---|
| 1 | $\frac{1}{10}$ | .1 | .1 | –2 | 4 | .4 |
| 2 | $\frac{2}{10}$ | .2 | .4 | –1 | 1 | .2 |
| 3 | $\frac{4}{10}$ | .4 | 1.2 | 0 | 0 | 0 |
| 4 | $\frac{2}{10}$ | .2 | .8 | 1 | 1 | .2 |
| 5 | $\frac{1}{10}$ | .1 | .5 | 2 | 4 | .4 |
| *Total* | $\frac{10}{10}$ | 1 | 3.0 | | | 1.2 |

$$\sum Y_i \bullet p(Y_i) = (1 * .1) + (2 * .2) + (3 * .4) + (4 * .2) + (5 * .1) = 3 = E(Y)$$

$$\sum [Y_i - E(Y)]^2 \bullet p(Y_i) = (4 * .1) + (1 * .2) + (0 * .4) + (1 * .2) + (4 * .1) = 1.2 = E[Y_i - E(Y)]^2$$

# II.Expected Values of Sum of Random Variables

let $Z_j = X_j + Y_j$

$E(Z_j) = E(X_j + Y_j) = \sum_x \sum_y (X_j + Y_j) p(X_j, Y_j) = \sum_x \sum_y (X_j) p(X_j Y_j) + \sum_x \sum_y Y_j p(X_j Y_j)$

Consider $\sum_x \sum_y (X_j) p(X_j Y_j)$ =recall evaluating double summations that summing the joint probability $p(X_j Y_j)$ over $Y_j$ yields the marginal probability of $X_j, p(X_j)$ (Summing down columns or across rows)

So summing over Y, $\sum_x (X_j) p(X_j, Y_j) = \sum_x (X_j) p(X_j)$

likewise for $\sum_x (Y_j) p(X_j, Y_j)$ summing over $X_j$ gives $\sum_x (Y_j) p(X_j, Y_j) = \sum_x (Y_j) p(Y_j)$

Thus

$E(Z_j) = E(X_j + Y_j) = \sum_x (X_j) p(X_j) + \sum_x (Y_j) p(Y_j)$ by definition of E( )

# III. Variance of Sum of Random variables

$$\textit{Let } Z = X_j + Y_j \qquad E(Z_j) = E(X_j + Y_j)$$

Var Z=var$(X_j + Y_j) = \sum_x \sum_y [(X_j + Y_j) - E(X_j + Y_j)]^2 p(X_j Y_j)$
using $E(X_j + Y_j) = E(x) + E(y)$

$$\begin{aligned} var(X_j + Y_j) &= \sum_x \sum_y [(X_j + Y_j) - E(X_j + Y_j)]^2 p(X_j Y_j) \\ &= \sum_x \sum_y [(X_j - E(X_j)) \ + (Y_j - E(Y_j))]^2 p(X_j Y_j) \\ &= \sum_x \sum_y [(X_j - E(X_j))^2 + 2(X_j - E(X_j)(Y_j - E(Y_j) + (Y_j - E(Y_j))^2] p(X_j Y_j) \end{aligned}$$

Recalling that

$$\sum_x \sum_y (X_j - E(X_j))^2 p(X_j Y_j) = \sum_x (X_j - E(X_j))^2 p(X_j) = \sigma_x^2$$

$$\sum_x \sum_y (Y_j - E(Y_j))^2 p(X_j Y_j) = \sum_x (Y_j - E(Y_j)))^2 p(Y_j) = \sigma_y^2$$

$$\sum_x \sum_y (X_j - E(X_j))(Y_j - E(Y_j) p(X_j Y_j) = \sigma_{xy}$$

var$(X_j + Y_j) = \sigma_x^2 + \sigma_y^2 + 2\sigma_{xy}$
if x,y independent, $\sigma_{xy} = 0,$ and var $(X_j + Y_j) = \sigma_x^2 + \sigma_y^2$
Let Z=$X_j - Y_j, E(Z_j) = E(X_j - Y_j)$

$$\begin{aligned} var(X_j - Y_j) &= \sum_x \sum_y [(X_j - Y_j) - E(X_j - Y_j)]^2 p(X_j Y_j) \\ &= \sum_x \sum_y [(X_j - E(X_j)) \ - (Y_j - E(Y_j))]^2 p(X_j Y_j) \\ &= \sum_x \sum_y [(X_j - E(X_j))^2 - 2(X_j - E(X_j)(Y_j - E(Y_j) + (Y_j - E(Y_j))^2] p(X_j Y_j) \\ &= \sigma_x^2 + \sigma_y^2 - 2\sigma_{xy} \end{aligned}$$

if x,y independent, $\sigma_{xy} = 0,$ and var $(X_j - Y_j) = \sigma_x^2 + \sigma_y^2$